Calculus

B.Tech all branches · GATE Engineering Mathematics

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1. Limits and Continuity

- **Limit**: lim (x→a) f(x) = L means f(x) gets as close to L as we like when x gets close to a.
- The limit exists only if left limit = right limit.
- **Continuity** at x = a: f(a) is defined, the limit exists, and lim (x→a) f(x) = f(a).
- **Differentiable ⇒ continuous**, but continuous does not mean differentiable. Example: |x| at x = 0.

➤ Standard limits

Limit	Value
lim (x→0) sin x / x	1
lim (x→0) (1 - cos x) / x^2	1/2
lim (x→0) (e^x - 1) / x	1
lim (x→0) (a^x - 1) / x	ln a
lim (x→0) ln(1 + x) / x	1
lim (x→0) (1 + x)^(1/x)	e
lim (x→inf) (1 + 1/x)^x	e
lim (x→a) (x^n - a^n) / (x - a)	n a^(n-1)

➤ L'Hospital's rule

- For 0/0 or inf/inf forms: lim f/g = lim f'/g' (differentiate top and bottom separately).
- Other forms (0 x inf, inf - inf, 1^inf, 0^0, inf^0) must first be changed into 0/0 or inf/inf.

FORMULA 1^inf form: lim f^g = e^(lim g(f - 1)), when f → 1 and g → inf.

EXAMPLE lim (x→0) (x - sin x) / x^3. 0/0 → (1 - cos x)/3x^2 → sin x / 6x → cos x / 6 = 1/6.

2. Differentiation

➤ Standard derivatives

f(x)	f'(x)
x^n	n x^(n-1)
e^x	e^x
a^x	a^x ln a
ln x	1/x
sin x	cos x
cos x	-sin x
tan x	sec^2 x
sin^-1 x	1 / sqrt(1 - x^2)
tan^-1 x	1 / (1 + x^2)

➤ Rules

- Product: (uv)' = u'v + uv'. Quotient: (u/v)' = (u'v - uv') / v^2.
- Chain rule: d/dx f(g(x)) = f'(g(x)) g'(x).
- **Leibniz theorem** (nth derivative of a product): (uv)_n = Σ nCr u_(n-r) v_r, r = 0 to n.

➤ Mean value theorems

- **Rolle's theorem**: f continuous on [a, b], differentiable on (a, b), f(a) = f(b) ⇒ some c in (a, b) has f'(c) = 0.
- **Lagrange MVT**: same first two conditions ⇒ f'(c) = (f(b) - f(a)) / (b - a).
- **Cauchy MVT**: f'(c)/g'(c) = (f(b) - f(a)) / (g(b) - g(a)).

EXAMPLE Lagrange MVT for f(x) = x^2 on [1, 3]: f'(c) = 2c = (9 - 1)/2 = 4, so c = 2, which lies in (1, 3).

➤ Taylor and Maclaurin series

- Taylor about x = a: f(x) = f(a) + (x-a) f'(a) + (x-a)^2/2! f''(a) + ...
- Maclaurin = Taylor about a = 0.

Function	Maclaurin series
e^x	1 + x + x^2/2! + x^3/3! + ...
sin x	x - x^3/3! + x^5/5! - ...
cos x	1 - x^2/2! + x^4/4! - ...
ln(1 + x)	x - x^2/2 + x^3/3 - ... (-1 < x <= 1)
1/(1 - x)	1 + x + x^2 + ... (abs x < 1)

3. Partial Derivatives

- For f(x, y): ∂f/∂x = derivative w.r.t. x keeping y constant.
- Mixed partials are equal for smooth functions: f_xy = f_yx.
- **Total derivative**: if z = f(x, y), x = x(t), y = y(t), then dz/dt = (∂z/∂x)(dx/dt) + (∂z/∂y)(dy/dt).

➤ Euler's theorem on homogeneous functions

- f(x, y) is homogeneous of degree n if f(tx, ty) = t^n f(x, y).

FORMULA x ∂f/∂x + y ∂f/∂y = n f, and x^2 f_xx + 2xy f_xy + y^2 f_yy = n(n - 1) f.

EXAMPLE f = x^3 + y^3 is homogeneous of degree 3. x(3x^2) + y(3y^2) = 3(x^3 + y^3) = 3f. Correct.

➤ Jacobian

- J = ∂(u, v)/∂(x, y) = determinant of | u_x u_y ; v_x v_y |.
- For polar coordinates x = r cos θ, y = r sin θ: ∂(x, y)/∂(r, θ) = r.
- If J = 0, u and v are functionally dependent.

4. Maxima and Minima

➤ One variable

- Solve f'(x) = 0 to get stationary points.
- f''(x) < 0 ⇒ maximum. f''(x) > 0 ⇒ minimum. f''(x) = 0 ⇒ test higher derivatives.

➤ Two variables

- Solve f_x = 0 and f_y = 0. At each point find r = f_xx, s = f_xy, t = f_yy.

Condition	Result
rt - s^2 > 0 and r < 0	Maximum
rt - s^2 > 0 and r > 0	Minimum
rt - s^2 < 0	Saddle point
rt - s^2 = 0	Test fails (needs more study)

➤ Solved example

- f(x, y) = x^2 + y^2 - 4x + 6y.
- f_x = 2x - 4 = 0 ⇒ x = 2. f_y = 2y + 6 = 0 ⇒ y = -3.
- r = 2, s = 0, t = 2. rt - s^2 = 4 > 0 and r > 0 ⇒ **minimum**.
- Minimum value = 4 + 9 - 8 - 18 = -9 at (2, -3).

➤ Lagrange multipliers

- To find extreme of f(x, y, z) with constraint g(x, y, z) = 0:
  - Make F = f + λ g. Solve F_x = 0, F_y = 0, F_z = 0, g = 0.

> **TIP** In constraint problems with a symmetric f and g, the answer is usually at x = y = z. Check it first.

5. Integration

➤ Standard integrals

f(x)	∫ f(x) dx
x^n (n ≠ -1)	x^(n+1)/(n+1)
1/x	ln abs(x)
e^(ax)	e^(ax)/a
sin x	-cos x
cos x	sin x
sec^2 x	tan x
1/(a^2 + x^2)	(1/a) tan^-1(x/a)
1/sqrt(a^2 - x^2)	sin^-1(x/a)

- **By parts**: ∫ u dv = uv - ∫ v du. Choose u with ILATE (Inverse, Log, Algebraic, Trig, Exponential).

Properties of definite integrals

- ∫ (0 to a) f(x) dx = ∫ (0 to a) f(a − x) dx.
- ∫ (−a to a) f(x) dx = 2 ∫ (0 to a) f(x) dx if f is even; = 0 if f is odd.

Wallis' formula (0 to π/2)

FORMULA ∫ (0 to π/2) sin^n x dx = [(n−1)(n−3)…] / [n(n−2)…] × k, where k = π/2 if n is even, k = 1 if n is odd.

EXAMPLE ∫ (0 to π/2) sin^4 x dx = (3 × 1)/(4 × 2) × π/2 = 3π/16.

Beta and Gamma functions

- Γ(n) = ∫ (0 to inf) e^(−x) x^(n−1) dx. Γ(n + 1) = n Γ(n) = n! for whole numbers.
- Γ(1/2) = sqrt(π). Γ(1) = 1.
- B(m, n) = ∫ (0 to 1) x^(m−1) (1 − x)^(n−1) dx = Γ(m) Γ(n) / Γ(m + n).

Solved example (by parts)

- ∫ x e^x dx: u = x, dv = e^x dx. = x e^x − ∫ e^x dx = e^x (x − 1) + C.
- Check: d/dx [e^x (x − 1)] = e^x (x − 1) + e^x = x e^x. Correct.

6. Applications of Integration

- Area between curves = ∫ (a to b) [upper − lower] dx.
- Volume of revolution about x-axis = ∫ (a to b) π y^2 dx.
- Arc length = ∫ (a to b) sqrt(1 + (dy/dx)^2) dx.

EXAMPLE Area between y = x and y = x^2 from 0 to 1 = ∫ (x − x^2) dx = 1/2 − 1/3 = 1/6.

7. Multiple Integrals

- **Double integral** ∫∫ f(x, y) dx dy over a region R. Area of R = ∫∫ dx dy.
- **Change of order**: draw the region, then write new limits. Very common in exams.
- **Polar form**: dx dy = r dr dθ.
- **Triple integral**: volume = ∫∫∫ dx dy dz.
  - Cylindrical: dV = r dr dθ dz. Spherical: dV = r^2 sin θ dr dθ dφ.

➤ Solved example

- ∫ (0 to 1) ∫ (0 to x) xy dy dx.
- Inner: ∫ (0 to x) xy dy = x (x^2/2) = x^3/2.
- Outer: ∫ (0 to 1) x^3/2 dx = (1/2)(1/4) = 1/8.

EXAMPLE Area of a circle of radius a by polar form: ∫ (0 to 2π) ∫ (0 to a) r dr dθ = (a^2/2)(2π) = πa^2.

8. Vector Calculus Basics

- **Del operator**: ∇ = i ∂/∂x + j ∂/∂y + k ∂/∂z.

Operation	Acts on	Gives	Meaning
Gradient ∇φ	Scalar	Vector	Direction of fastest increase, normal to surface
Divergence ∇ . F	Vector	Scalar	Net outflow per volume
Curl ∇ x F	Vector	Vector	Rotation of the field

- div F = 0 ⇒ **solenoidal**. curl F = 0 ⇒ **irrotational** (F = ∇φ for some φ).
- Identities: curl (grad φ) = 0, div (curl F) = 0.
- **Directional derivative** of φ along unit vector a = ∇φ . a.

EXAMPLE φ = x^2 y + z at (1, 2, 0): ∇φ = (2xy, x^2, 1) = (4, 1, 1). Maximum rate of change = sqrt(16 + 1 + 1) = sqrt(18) = 3 sqrt(2).

➤ The three integral theorems

Theorem	Connects
Green's	Line integral round closed curve C <–> double integral over plane region
Stokes'	∮ F . dr = ∫∫ (curl F) . n dS
Gauss divergence	∫∫ F . n dS (closed surface) = ∫∫∫ div F dV

FORMULA Green's: ∮ (M dx + N dy) = ∫∫ (∂N/∂x - ∂M/∂y) dx dy.

9. Most Asked Questions (practice)

1. Evaluate lim (x→0) (1 - cos x)/x^2 using L'Hospital's rule.
2. State and verify Rolle's theorem for f(x) = x^2 - 5x + 6 on [2, 3].

3. Expand e^x sin x by Maclaurin series up to x^3.

4. Verify Euler's theorem for u = x^2 + xy + y^2.

5. Find the maxima and minima of f(x, y) = x^3 + y^3 - 3xy.

6. Find the extreme of x^2 + y^2 + z^2 subject to x + y + z = 3 (Lagrange).

7. Evaluate ∫ (0 to π/2) sin^5 x cos^2 x dx using Beta function.

8. Change the order of integration and evaluate ∫ (0 to 1) ∫ (x to 1) e^(y^2) dy dx.

9. Find div and curl of F = xy i + yz j + zx k.

10. Verify Green's theorem for a given field over the unit square.

TIP GATE asks limits, maxima/minima, definite integrals and gradient/directional derivative most. In university exams, change of order and the three vector theorems are sure-shot long questions.