Complex Analysis

B.Tech all branches · GATE Engineering Mathematics

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1. Complex Numbers

- $z = x + iy$, with $i^2 = -1$. $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.
- **Conjugate** $\bar{z} = x - iy$. **Modulus** $|z| = r = \sqrt{x^2 + y^2}$. $z\bar{z} = |z|^2$.
- **Polar form**: $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ (Euler's formula). $\theta = \arg z$.
- Product: moduli multiply, arguments add. Quotient: moduli divide, arguments subtract.

FORMULA De Moivre: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

➤ nth roots

- $z^{1/n} = r^{1/n} e^{i(\theta + 2k\pi)/n}$, $k = 0, 1, \ldots, n-1$. There are exactly n roots, equally spaced on a circle.

EXAMPLE Cube roots of 1: $e^{i\,2k\pi/3} = 1, \omega, \omega^2$ with $\omega = (-1 + i\sqrt{3})/2$. Sum $1 + \omega + \omega^2 = 0$.

➤ Standard complex functions

Function	Definition / fact
e^z	$e^x(\cos y + i\sin y)$, period $2\pi i$
$\sin z$	$(e^{iz} - e^{-iz})/2i$
$\cos z$	$(e^{iz} + e^{-iz})/2$
$\log z$	$\ln r + i(\theta + 2k\pi)$, many valued
$\sinh z$	$(e^z - e^{-z})/2$, $\sin(iz) = i\sinh z$

REMEMBER $|\sin z|$ and $|\cos z|$ are NOT bounded by 1 in the complex plane (they grow with $|y|$).

2. Analytic Functions

- $f(z) = u(x, y) + i\, v(x, y)$.
- f is **analytic** at z_0 if it is differentiable at every point near z_0.
- **Entire** function: analytic everywhere (e^z, $\sin z$, polynomials).

➤ Cauchy–Riemann (CR) equations

FORMULA $u_x = v_y$ and $u_y = -v_x$. Polar form: $u_r = (1/r)\, v_\theta$, $v_r = -(1/r)\, u_\theta$.

- CR equations + continuous partial derivatives ⇒ f is analytic.
- Then $f'(z) = u_x + i\, v_x$.
- $f(z) = \bar{z}$ is nowhere analytic ($u_x = 1$, $v_y = -1$).

➤ Harmonic functions

- u is **harmonic** if $u_{xx} + u_{yy} = 0$ (Laplace equation).
- Real and imaginary parts of an analytic function are both harmonic. v is the **harmonic conjugate** of u.

➤ Solved example

- $u = x^2 - y^2$. $u_{xx} + u_{yy} = 2 - 2 = 0$, so u is harmonic.
- $v_y = u_x = 2x \Rightarrow v = 2xy + g(x)$. $v_x = -u_y = 2y \Rightarrow 2y + g'(x) = 2y \Rightarrow g = \text{const}$.
- $v = 2xy$, $f(z) = (x + iy)^2 = z^2$.

TIP Milne–Thomson method: to get $f(z)$ from u, find $f'(z) = u_x(z, 0) - i\, u_y(z, 0)$ and integrate. Put $x = z$, $y = 0$.

3. Conformal Mapping

- A map $w = f(z)$ is **conformal** where f is analytic and $f'(z) \neq 0$: it keeps angles (size and direction).
- Points where $f'(z) = 0$ are **critical points**.

Map	Effect
$w = z + c$	Translation
$w = az$, a real	Scaling (magnification)
$w = e^{i\alpha} z$	Rotation by α
$w = 1/z$	Inversion + reflection; circles/lines go to circles/lines
$w = z^2$	Doubles angles at origin

➤ Bilinear (Mobius) transformation

- $w = (az + b)/(cz + d)$, with $ad - bc \neq 0$.
- Maps circles and lines to circles and lines. Fixed by the images of **3 points**.
- **Fixed points**: solve $z = (az + b)/(cz + d)$.
- Cross-ratio is kept: $(w1, w2, w3, w) = (z1, z2, z3, z)$.

EXAMPLE Fixed points of $w = (z - 1)/(z + 1)$: $z^2 + z = z - 1 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$.

4. Complex Integration

- Line integral $\int_C f(z)\, dz$ along a curve C. Use $z = z(t)$, $dz = z'(t)\, dt$.
- **Cauchy's theorem**: f analytic inside and on a closed curve $C \Rightarrow \oint f(z)\, dz = 0$.

FORMULA Cauchy integral formula: $\oint f(z)/(z - a)\, dz = 2\pi i\, f(a)$, for a inside C.

FORMULA Derivatives: $\oint f(z)/(z - a)^{n+1}\, dz = (2\pi i / n!)\, f^{(n)}(a)$.

➤ Solved example

- $\oint e^z / (z - 1)\, dz$ over $|z| = 2$.
- $z = 1$ is inside. $f(z) = e^z$. Answer $= 2\pi i\, e^1 = 2\pi i\, e$.

EXAMPLE $\oint dz/z$ over $|z| = 1 = 2\pi i$ ($f = 1$, $a = 0$). This is the basic result to remember.

➤ Solved example (derivative formula)

- $\oint e^{2z} / (z - 1)^3\, dz$ over $|z| = 2$. Here $n = 2$, $a = 1$, $f = e^{2z}$.
- $f''(z) = 4e^{2z}$, so $f''(1) = 4e^2$.
- Integral $= (2\pi i / 2!) \cdot 4e^2 = 4\pi i\, e^2$.
- **Morera's theorem**: converse of Cauchy's theorem.

- **Liouville's theorem**: a bounded entire function is constant.

5. Series

- **Taylor series** about a (f analytic in abs(z - a) < R): f(z) = Σ f^(n)(a) (z - a)^n / n!.
- **Laurent series** in an annulus r < abs(z - a) < R: f(z) = Σ an (z - a)^n + Σ bn (z - a)^(-n).
  - The part with negative powers is the **principal part**.

➤ Solved example (Laurent)

- f(z) = 1/(z(z - 1)) for 0 < abs(z) < 1.
- 1/(z - 1) = -1/(1 - z) = -(1 + z + z^2 + ...).
- f = (1/z)(-(1 + z + z^2 + ...)) = -1/z - 1 - z - z^2 - ....
- Coefficient of 1/z = -1, which is the residue at z = 0.

6. Singularities and Residues

- **Singularity**: point where f is not analytic. **Isolated** if f is analytic in a punctured disc around it.

Type	Laurent principal part	Example
Removable	No negative powers	sin z / z at 0
Pole of order m	Finite, ends at (z - a)^(-m)	1/(z - 2)^3 at 2 (order 3)
Essential	Infinite negative powers	e^(1/z) at 0

- **Zero** of order m: f(a) = ... = f^(m-1)(a) = 0, f^(m)(a) ≠ 0.
- **Residue** at a = coefficient b1 of 1/(z - a) in the Laurent series.

➤ Residue formulas

- Simple pole: Res = lim (z→a) (z - a) f(z).
- If f = P/Q with simple zero of Q at a: Res = P(a) / Q'(a).
- Pole of order m: Res = (1/(m-1)!) lim (z→a) d^(m-1)/dz^(m-1) [(z - a)^m f(z)].

FORMULA Residue theorem: ∮_C f(z) dz = 2πi x (sum of residues inside C).

➤ Solved example

- ∮ (z + 1) / (z(z - 2)) dz over abs(z) = 3. Both poles 0 and 2 are inside.
- Res at 0 = (0 + 1)/(0 - 2) = -1/2. Res at 2 = (2 + 1)/2 = 3/2.
- Integral = 2πi (-1/2 + 3/2) = 2πi.

7. Real Integrals by Residues

- **Type 1**: ∫ (0 to 2π) F(cos θ, sin θ) dθ. Put z = e^(iθ), cos θ = (z + 1/z)/2, sin θ = (z - 1/z)/2i, dθ = dz/(iz). Integrate on abs(z) = 1.
- **Type 2**: ∫ (-inf to inf) P(x)/Q(x) dx (degree Q >= degree P + 2, no real poles) = 2πi x (sum of residues in the upper half plane).

EXAMPLE ∫ (-inf to inf) dx/(1 + x^2). Pole z = i in upper half. Res = 1/(2i). Integral = 2πi/(2i) = π.

➤ Solved example (Type 1)

- I = ∫ (0 to 2π) dθ / (5 + 4 cos θ).
- Put z = e^(iθ): 5 + 4 cos θ = 5 + 2(z + 1/z) = (2z^2 + 5z + 2)/z. dθ = dz/(iz).
- I = ∮ dz / (i(2z^2 + 5z + 2)) = ∮ dz / (i(2z + 1)(z + 2)).
- Poles: z = -1/2 (inside abs(z) = 1) and z = -2 (outside).
- Res at -1/2 = 1 / (i . 2 . (-1/2 + 2)) = 1 / (3i).
- I = 2πi . 1/(3i) = 2π/3.

FORMULA Check: ∫ (0 to 2π) dθ/(a + b cos θ) = 2π / sqrt(a^2 - b^2) for a > abs(b). Here 2π/sqrt(25 - 16) = 2π/3.

➤ Jordan's lemma (Type 3)

- For ∫ (-inf to inf) f(x) e^(imx) dx with m > 0 and f → 0 at infinity, the big semicircle part goes to 0.
- Take the real or imaginary part at the end to get the cos mx or sin mx integral.

8. Most Asked Questions (practice)

1. Show that u = e^x cos y is harmonic and find the analytic function f(z).
2. Prove that an analytic function with constant modulus is constant.

3. State and derive the Cauchy–Riemann equations in Cartesian and polar form.

4. Find the bilinear map sending $z = 0, 1, \infty$ to $w = -1, -i, 1$.

5. Evaluate $\oint (\sin \pi z^2 + \cos \pi z^2) / ((z - 1)(z - 2))\, dz$ over $|z| = 3$.

6. Expand $1/((z - 1)(z - 2))$ in Laurent series for $1 < |z| < 2$.

7. Classify the singularities of $(1 - \cos z)/z^2$ and $e^{(1/z)}$.

8. Find residues of $z^2 / ((z - 1)^2 (z + 2))$ at its poles.

9. Evaluate $\int_0^{2\pi} d\theta / (2 + \cos \theta)$ by contour integration.

10. State Cauchy's integral formula and Liouville's theorem.

TIP GATE mostly asks CR equations, residue values and contour integrals by the residue theorem. Always check which poles lie inside the contour first.