Control Systems

Short notes for B.Tech EE · EEE · ECE · GATE EE / EC

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I. Introduction

- **Control system** = a set of components that controls the output to get a desired response.
- **Open loop**: output has no effect on the control action. Example: washing machine timer, traffic signal with fixed timing.
- **Closed loop (feedback)**: output is measured and compared with the input. Example: AC with thermostat, voltage regulator.

Point	Open loop	Closed loop
Feedback	No	Yes
Accuracy	Low	High
Effect of disturbance	Large	Reduced
Stability	Generally stable	Can become unstable
Cost / complexity	Low	High

- **Transfer function (TF)** = Laplace of output / Laplace of input, with all initial conditions zero.
- TF is defined only for **LTI** (linear time-invariant) systems.
- **Poles**: roots of the denominator. **Zeros**: roots of the numerator.
- **Order** = highest power of s in the denominator. **Type** = number of poles at the origin in the open-loop G(s)H(s).

FORMULA Closed loop TF = G / (1 + G H) for negative feedback; G / (1 - G H) for positive feedback.

FORMULA Sensitivity of closed-loop T to G: S = 1 / (1 + G H). Feedback reduces sensitivity to parameter changes.

2. Block Diagrams and Signal Flow Graphs

➤ Block diagram rules

- Blocks in **series**: multiply (G1 G2). Blocks in **parallel**: add (G1 + G2).
- Feedback loop: G / (1 + G H).
- Moving a take-off point or summing point across a block: multiply or divide by that block's TF to keep signals the same.

➤ Signal flow graph (SFG)

- Nodes = variables, branches = gains. **Forward path**: from input to output, no node twice.
- **Loop**: closed path. **Non-touching loops**: loops with no common node.

FORMULA Mason's gain formula: $T = (\Sigma\ Pk\ \Delta k) / \Delta$. Δ = 1 - (sum of all loop gains) + (sum of products of 2 non-touching loops) - (3 non-touching) + ...

- Δk = value of Δ for the part of the graph not touching the k-th forward path.

EXAM POINT Mason's formula questions come almost every year. Count loops and non-touching pairs carefully; missing one pair is the usual mistake.

➤ Mechanical-electrical analogy

Mechanical (translation)	Force-voltage	Force-current
Force F	Voltage V	Current I
Mass M	Inductance L	Capacitance C
Damper B	Resistance R	Conductance 1/R
Spring K	1/C	1/L
Velocity	Current	Voltage

3. Time Response

- Standard test inputs: step, ramp, parabolic, impulse. Impulse response = inverse Laplace of TF.

- **First-order** system $1/(sT + 1)$: step response $c(t) = 1 - e^{(-t/T)}$. At $t = T$ output is 63.2%. Settling time (2%) = $4T$.

➤ Second-order system

- Standard form: $\omega n^2 / (s^2 + 2\zeta\omega n\, s + \omega n^2)$. ζ = damping ratio, ωn = natural frequency.

ζ	Type	Poles
$\zeta = 0$	Undamped (sustained oscillation)	On $j\omega$ axis
$0 < \zeta < 1$	Underdamped	Complex, left half
$\zeta = 1$	Critically damped	Real, equal
$\zeta > 1$	Overdamped	Real, different

FORMULA $\omega d = \omega n \sqrt{1 - \zeta^2}$. $\theta = \cos^{-1}(\zeta)$. Rise time $Tr = (\pi - \theta)/\omega d$. Peak time $Tp = \pi/\omega d$.

FORMULA Max overshoot $Mp = e^{(-\zeta\pi / \sqrt{1 - \zeta^2})}$. Settling time $Ts = 4/(\zeta\omega n)$ for 2%, $3/(\zeta\omega n)$ for 5%.

- Mp depends **only on** ζ. Larger $\zeta \rightarrow$ less overshoot but slower rise.

EXAMPLE $G(s) = 25 / (s(s + 6))$, unity feedback. $CLTF = 25/(s^2 + 6s + 25)$. So $\omega n = 5$ rad/s, $2\zeta\omega n = 6$, $\zeta = 0.6$. $\omega d = 5 \times 0.8 = 4$ rad/s. $Tp = \pi/4 = 0.785$ s. $Mp = e^{(-0.6\pi/0.8)} = e^{(-2.356)} = 0.095 = 9.5\%$. Ts (2%) $= 4/3 = 1.33$ s. $\theta = 0.927$ rad, $Tr = (3.142 - 0.927)/4 = 0.554$ s.

4. Steady-State Error

- For unity feedback: $ess = \lim_{(s\to 0)}\, s\, R(s) / (1 + G(s))$.

FORMULA $Kp = \lim G(s)$, $Kv = \lim s\, G(s)$, $Ka = \lim s^2\, G(s)$ (all as $s \rightarrow 0$).

Type	Step (1/s)	Ramp ($1/s^2$)	Parabolic ($1/s^3$)
0	$1/(1 + Kp)$	∞	∞
1	0	$1/Kv$	∞
2	0	0	$1/Ka$

EXAMPLE $G(s) = 10 / (s(s + 2))$, unit ramp input. Type 1. $K_v = \lim s \cdot 10/(s(s+2)) = 10/2 = 5$. $e_{ss} = 1/K_v = 0.2$.

REMEMBER Higher type → less steady-state error, but stability gets worse.

5. Stability and Routh-Hurwitz

- A system is **stable** if all closed-loop poles lie in the left half of the s-plane.
- Poles on the $j\omega$ axis (non-repeated) → **marginally stable**. Any pole in the right half → unstable.
- **Necessary condition**: all coefficients of the characteristic equation present and of the same sign.

➤ Routh array

- First two rows from the coefficients. Next rows by the cross-multiplication rule.
- Number of sign changes in the first column = number of poles in the right half plane.
- **Special case 1**: zero in first column → replace it with a small ε.
- **Special case 2**: whole row zero → make the auxiliary equation $A(s)$ from the row above, use dA/ds. Roots of $A(s)$ are symmetric poles.

EXAMPLE $G(s)H(s) = K / (s(s + 1)(s + 2))$. Char. eq: $s^3 + 3s^2 + 2s + K = 0$. Rows: s^3: 1, 2 | s^2: 3, K | s^1: $(6 - K)/3$ | s^0: K. Stable for $0 < K < 6$. At $K = 6$: $3s^2 + 6 = 0$, $s = \pm j\sqrt{2}$, so it oscillates at $\omega = \sqrt{2}$ rad/s.

6. Root Locus

- Path of closed-loop poles as K changes from 0 to ∞.

➤ Rules

1. Locus starts at open-loop **poles** ($K = 0$) and ends at **zeros** or at infinity ($K = \infty$).
2. Number of branches = number of open-loop poles (P).
3. A point on the real axis is on the locus if the total number of poles + zeros to its **right is odd**.
4. Asymptote angles = $(2q + 1)\ 180° / (P - Z)$.
5. Centroid = $(\Sigma$ real parts of poles $- \Sigma$ real parts of zeros$) / (P - Z)$.

6. Breakaway point: solve dK/ds = 0.

7. Crossing of jω axis: from the Routh array.

8. Symmetric about the real axis.

EXAMPLE For K/(s(s+1)(s+2)): centroid = (0 - 1 - 2)/3 = -1. Angles = 60°, 180°, 300°. K = -(s^3 + 3s^2 + 2s), dK/ds = 0 gives 3s^2 + 6s + 2 = 0, s = -0.423 (valid, between 0 and -1) or -1.577 (not on locus).

- Adding a pole pulls the locus to the right (less stable). Adding a zero pulls it to the left (more stable).

7. Frequency Response

- Put s = jω. Study |G(jω)| and ∠G(jω) as ω changes.
- **Resonant peak** Mr = 1 / (2ζ sqrt(1 - ζ^2)), valid for ζ < 0.707. **Resonant frequency** ωr = ωn sqrt(1 - 2ζ^2).
- **Bandwidth**: frequency at which closed-loop gain falls to 0.707 (-3 dB) of the DC value.

➤ Bode plot

Factor	Magnitude slope	Phase
K	Flat line at 20 log K	0°
Pole at origin 1/s	-20 dB/decade	-90°
Zero at origin s	+20 dB/decade	+90°
Simple pole 1/(1 + sT)	-20 dB/dec after ω = 1/T	0 to -90°
Simple zero (1 + sT)	+20 dB/dec after ω = 1/T	0 to +90°
Quadratic pole	-40 dB/dec after ωn	0 to -180°

- **Gain crossover frequency** ωgc: |G H| = 1 (0 dB). **Phase crossover frequency** ωpc: ∠G H = -180°.

FORMULA Gain margin GM = 1 / |G H(jωpc)| (in dB: -20 log |G H(jωpc)|).
Phase margin PM = 180° + ∠G H(jωgc).

- Stable (minimum-phase system) when both GM and PM are positive, i.e. ωgc < ωpc.

Nyquist criterion

- Plot G H(jω) for ω from -∞ to +∞ and count encirclements of the point **-1 + j0**.
- Z = P - N, where P = open-loop poles in right half plane, N = number of anticlockwise encirclements of -1 + j0, Z = closed-loop poles in right half plane.
- Closed loop is stable when Z = 0.

8. Compensators and Controllers

Compensator	TF form	Effect
Lead	(1 + aTs)/(1 + Ts), a > 1	Adds phase lead, improves transient response, increases bandwidth
Lag	(1 + Ts)/(1 + bTs), b > 1	Improves steady-state error, reduces bandwidth
Lag-lead	Both	Improves both

- Lead = like a high-pass filter (PD-type). Lag = like a low-pass filter (PI-type).
- Maximum phase lead ϕm: sin ϕm = (a - 1)/(a + 1).

PID controller

Action	Effect
P (proportional)	Faster response, reduces but does not remove ess
I (integral)	Removes steady-state error (raises type by 1), may increase overshoot
D (derivative)	Adds damping, reduces overshoot, sensitive to noise

- Output u(t) = Kp e + Ki ∫e dt + Kd de/dt.

9. State Space Analysis

- State equations: $\dot{x} = A\,x + B\,u$, output $y = C\,x + D\,u$.
- Can handle MIMO, non-zero initial conditions and time-varying systems.

FORMULA Transfer function from state model: G(s) = C (sI - A)^-1 B + D.
Characteristic equation: |sI - A| = 0 (eigenvalues of A = poles).

FORMULA State transition matrix ϕ(t) = e^(At) = inverse Laplace of (sI - A)^-1.
ϕ(0) = I.

- **Controllability** (Kalman): Qc = [B AB A^2B ... A^(n-1)B] must have rank n.
- **Observability**: Qo = [C ; CA ; CA^2 ; ... ; CA^(n-1)] must have rank n.
- Simple check: rank n means the determinant of Qc (or Qo) is not zero for a single-input or single-output system.

TIP For a 2x2 matrix A = [a b ; c d], (sI - A)^-1 = (1/Δ) [s-d b ; c s-a], where Δ = s^2 - (a + d)s + (ad - bc).

10. Most Asked Questions (practice)

1. Compare open-loop and closed-loop systems. Show how feedback reduces sensitivity.
2. Reduce a block diagram and verify the result with Mason's gain formula.
3. Derive the step response of an underdamped second-order system. Find Tr, Tp, Mp, Ts.
4. Find steady-state error for step, ramp and parabolic inputs; define Kp, Kv, Ka.
5. Using Routh-Hurwitz, find the range of K for stability and the frequency of oscillation.
6. Sketch the root locus of K/(s(s+1)(s+2)) with all the rules.
7. Draw the Bode plot and find gain margin and phase margin.
8. State and apply the Nyquist stability criterion.
9. Design a lead compensator; compare lead and lag compensators.
10. Find the TF from a state model and test controllability and observability.

TIP For GATE EE, master second-order specs, Routh, root locus rules, GM/PM from Bode and Nyquist encirclements. Most numericals are short if you know these formulas well.