Differential Equations

B.Tech all branches · GATE Engineering Mathematics

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1. Basics

- **Differential equation (DE)** = an equation with derivatives of an unknown function.
- **ODE**: one independent variable. **PDE**: two or more independent variables.
- **Order** = highest derivative present. **Degree** = power of the highest derivative (after removing radicals and fractions).
- **Linear**: y and its derivatives appear only to power 1 and are not multiplied together.
- **General solution** has as many arbitrary constants as the order. **Particular solution** = constants fixed by conditions.

EXAMPLE $(d^2y/dx^2)^3 + (dy/dx)^5 + y = 0$ has order 2, degree 3. It is non-linear.

2. First Order, First Degree ODEs

Type	Form	Method
Variable separable	$dy/dx = f(x)\,g(y)$	$\int dy/g(y) = \int f(x)\,dx$
Homogeneous	$dy/dx = f(y/x)$	Put $y = vx$, $dy/dx = v + x\,dv/dx$
Linear	$dy/dx + P\,y = Q$	Integrating factor $IF = e^{\int P\,dx}$
Bernoulli	$dy/dx + P\,y = Q\,y^n$	Divide by y^n, put $z = y^{1-n}$
Exact	$M\,dx + N\,dy = 0$	Exact if $\partial M/\partial y = \partial N/\partial x$

FORMULA Linear DE solution: $y \cdot IF = \int Q \cdot IF\,dx + C$, where $IF = e^{\int P\,dx}$.

➤ Solved example (variable separable)

- $dy/dx = xy$, $y(0) = 1$.
- $dy/y = x\,dx \Rightarrow \ln y = x^2/2 + C$.
- $y(0) = 1 \Rightarrow C = 0$. So $y = e^{x^2/2}$.

Solved example (linear)

- $dy/dx + y = e^x$. $P = 1$, $Q = e^x$. $IF = e^x$.
- $y\,e^x = \int e^{2x}\,dx = e^{2x}/2 + C$.
- $y = e^x/2 + C\,e^{-x}$.
- Check: $y' = e^x/2 - C\,e^{-x}$. $y' + y = e^x$. Correct.

Exact equations

- Solution: $\int M\,dx$ (treat y as constant) $+ \int$ (terms of N without x) $dy = C$.
- If not exact, look for an integrating factor. Example: if $(M_y - N_x)/N = f(x)$ only, $IF = e^{\int f(x)\,dx}$.

Solved example (exact)

- $(2xy + 3)\,dx + (x^2 + 4y)\,dy = 0$. $M = 2xy + 3$, $N = x^2 + 4y$.
- $\partial M/\partial y = 2x = \partial N/\partial x$, so it is exact.
- $\int M\,dx = x^2 y + 3x$. Terms of N without x: $4y$, $\int 4y\,dy = 2y^2$.
- $x^2 y + 3x + 2y^2 = C$.

First order, higher degree ($p = dy/dx$)

- **Solvable for p**: factorise into $(p - f1)(p - f2) = 0$ and solve each.
- **Clairaut's equation**: $y = px + f(p)$. General solution: replace p by c, so $y = cx + f(c)$ (a family of straight lines).
- The singular solution comes by removing c from $y = cx + f(c)$ and $0 = x + f'(c)$.

EXAMPLE $y = px + p^2$. General: $y = cx + c^2$. From $x + 2c = 0$, $c = -x/2$, so the singular solution is $y = -x^2/4$.

TIP Before any method, first check if variables can be separated. It is the quickest route.

3. Applications of First Order ODEs

- **Growth / decay**: $dN/dt = kN \Rightarrow N = N0\,e^{kt}$. $k > 0$ growth, $k < 0$ decay.
- **Half-life**: $T = \ln 2 / |k|$.
- **Newton's law of cooling**: $dT/dt = -k\,(T - Ts) \Rightarrow T = Ts + (T0 - Ts)\,e^{-kt}$.
- **RL circuit**: $L\,di/dt + R\,i = E \Rightarrow i = (E/R)(1 - e^{-Rt/L})$ when $i(0) = 0$.
- **Orthogonal trajectories**: replace dy/dx by $-dx/dy$ in the DE of the family.

EXAMPLE A substance halves in 10 years. $k = \ln 2 / 10$. After 30 years, amount left $= (1/2)^3 = 1/8$ of the start.

4. Higher Order Linear ODEs (constant coefficients)

- Form: $a\,y'' + b\,y' + c\,y = f(x)$. Write $D = d/dx$.
- **Complete solution** $y = CF + PI$ (Complementary Function + Particular Integral).
- **Auxiliary equation (AE)**: $a\,m^2 + b\,m + c = 0$.

➤ Complementary function

Roots of AE	CF
Real, distinct $m1, m2$	$c1\,e^{(m1\,x)} + c2\,e^{(m2\,x)}$
Real, equal m, m	$(c1 + c2\,x)\,e^{(mx)}$
Complex $\alpha \pm i\beta$	$e^{(\alpha x)}(c1 \cos \beta x + c2 \sin \beta x)$

➤ Particular integral PI = f(x) / F(D)

f(x)	Rule
$e^{(ax)}$	$e^{(ax)} / F(a)$, if $F(a) \neq 0$. If $F(a) = 0$: $x\,e^{(ax)} / F'(a)$
sin ax or cos ax	Put $D^2 = -a^2$ in $F(D)$. If it gives 0, multiply by x and differentiate F
x^n	Expand $[F(D)]^{-1}$ as a power series in D up to D^n
$e^{(ax)}\,V(x)$	$e^{(ax)} \cdot [1 / F(D + a)]\,V$

➤ Solved example

- $y'' - 5y' + 6y = e^{(4x)}$.
- AE: $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$. $CF = c1\,e^{(2x)} + c2\,e^{(3x)}$.
- $PI = e^{(4x)} / (16 - 20 + 6) = e^{(4x)} / 2$.
- $y = c1\,e^{(2x)} + c2\,e^{(3x)} + e^{(4x)}/2$.

➤ Solved example (complex roots)

- $y'' + 4y = 0$. AE: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$.
- $y = c1 \cos 2x + c2 \sin 2x$.

EXAMPLE $y'' + y = \sin x$. $D^2 = -1$ makes $F = 0$, so $PI = x \cdot (1/(2D)) \sin x = (x/2)(-\cos x) = -x \cos x / 2$.

Applications of second order ODEs

System	Equation	Note
Spring-mass (free)	m x'' + k x = 0	SHM, ω = sqrt(k/m)
Damped spring	m x'' + c x' + k x = 0	Over / critical / under damped by c^2 - 4mk
LCR circuit	L q'' + R q' + q/C = E(t)	Same form as damped spring

- Under damped (c^2 < 4mk): oscillation dies out. Critical (c^2 = 4mk): fastest return without oscillation.
- **Resonance**: forcing frequency = natural frequency gives a growing x sin(ωt) term in the PI.

5. Cauchy–Euler and Variation of Parameters

Cauchy–Euler equation

- Form: x^2 y'' + a x y' + b y = f(x).
- Put x = e^z, θ = d/dz. Then x y' = θy and x^2 y'' = θ(θ - 1) y.
- It becomes a constant coefficient ODE in z.

EXAMPLE x^2 y'' - 2y = 0. θ(θ - 1) - 2 = 0 ⇒ θ = 2, -1. y = c1 x^2 + c2 x^(-1).

Variation of parameters

- For y'' + P y' + Q y = R with CF = c1 y1 + c2 y2.
- **Wronskian** W = y1 y2' - y2 y1'.

FORMULA PI = -y1 ∫ (y2 R / W) dx + y2 ∫ (y1 R / W) dx.

- Use it when R is like sec x, tan x, cosec x (the short-cut rules fail).
- y1, y2 are independent ⟺ W ≠ 0.

6. Series Solutions and Special Functions

- Near an **ordinary point** use $y = \Sigma\, a_n\, x^n$. Near a **regular singular point** use **Frobenius**: $y = \Sigma\, a_n\, x^{(n+r)}$.
- **Legendre equation**: $(1 - x^2)\, y'' - 2x\, y' + n(n + 1)\, y = 0$. Solutions: Legendre polynomials $P_n(x)$.
  - $P_0 = 1$, $P_1 = x$, $P_2 = (3x^2 - 1)/2$.
  - Orthogonal on $[-1, 1]$: $\int P_m P_n\, dx = 0$ $(m \neq n)$, $= 2/(2n + 1)$ $(m = n)$.
- **Bessel equation**: $x^2\, y'' + x\, y' + (x^2 - n^2)\, y = 0$. Solutions: $J_n(x)$.

> **REMEMBER** Rodrigues formula: $P_n(x) = (1 / (2^n\, n!))\, d^n/dx^n\, (x^2 - 1)^n$.

7. Partial Differential Equations

- **Lagrange linear PDE**: $P\, p + Q\, q = R$, where $p = \partial z/\partial x$, $q = \partial z/\partial y$.
  - Auxiliary equations: $dx/P = dy/Q = dz/R$. Find two solutions $u = a$, $v = b$. Answer: $\phi(u, v) = 0$.
- Second order PDE $A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + \ldots = 0$. Classification by $B^2 - 4AC$:

$B^2 - 4AC$	Type	Example
< 0	Elliptic	Laplace: $u_{xx} + u_{yy} = 0$
$= 0$	Parabolic	Heat: $u_t = c^2\, u_{xx}$
> 0	Hyperbolic	Wave: $u_{tt} = c^2\, u_{xx}$

➤ Separation of variables

- Assume $u(x, t) = X(x)\, T(t)$. Put into the PDE and split into two ODEs with a separation constant $-k^2$.
- Heat equation (rod length L, ends at 0 °C): $u = \Sigma\, b_n \sin(n\pi x/L)\, e^{(-c^2\, n^2\, \pi^2\, t\, /\, L^2)}$.
- Wave equation (string fixed at ends): $u = \Sigma \sin(n\pi x/L)\, [a_n \cos(n\pi ct/L) + b_n \sin(n\pi ct/L)]$.

> **TIP** Choose the negative constant $(-k^2)$ for heat and wave problems. It gives decaying or oscillating solutions that match the physics.

8. Most Asked Questions (practice)

1. Solve $(x + y)\,dx + (x - y)\,dy = 0$ (check exactness).
2. Solve $dy/dx + y \tan x = \sec x$.
3. Solve the Bernoulli equation $dy/dx + y = x\,y^3$.
4. Solve $y'' + 2y' + y = x\,e^{-x}$.
5. Solve $y'' + 4y = \sin 2x$.
6. Solve $y'' + y = \sec x$ by variation of parameters.
7. Solve $x^2\,y'' + x\,y' + y = \log x$.
8. Find the series solution of $y'' + x\,y = 0$ near $x = 0$.
9. Classify $u_{xx} + 4u_{xy} + 4u_{yy} = 0$ and solve the 1D heat equation for given end conditions.
10. Newton's law of cooling numerical: body cools from 80 °C to 60 °C in 20 min in a 20 °C room.

TIP GATE asks first order linear / exact ODEs, CF of constant-coefficient ODEs, Cauchy–Euler and PDE classification. In university exams, PI rules and variation of parameters are certain.