Discrete Mathematics

Short notes for B.Tech CSE / IT · BCA · MCA · GATE CS

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1. Sets and Functions

- **Set** = a well-defined collection of distinct objects. Example: A = {1, 2, 3}.
- Operations: union (A ∪ B), intersection (A ∩ B), difference (A − B), complement (A'), symmetric difference (A ⊕ B).
- **Power set** P(A) = set of all subsets. If |A| = n then |P(A)| = 2^n.
- **Cartesian product** A x B = all ordered pairs (a, b). |A x B| = |A| · |B|.

FORMULA Inclusion-exclusion: |A ∪ B| = |A| + |B| − |A ∩ B|. For 3 sets: |A ∪ B ∪ C| = |A| + |B| + |C| − |A∩B| − |B∩C| − |A∩C| + |A∩B∩C|.

EXAMPLE In 1 to 100, numbers divisible by 2 or 3 = 50 + 33 − 16 = 67.

➤ Functions

Type	Meaning
One-one (injective)	Different inputs give different outputs
Onto (surjective)	Every element of codomain is an output
Bijective	One-one and onto, so it has an inverse

- From A (|A| = m) to B (|B| = n): total functions = n^m. One-one functions = n!/(n − m)! if m <= n.
- Number of onto functions from m elements to 2 elements = 2^m − 2.

2. Relations

- **Relation** R on set A = a subset of A x A. Total relations on n elements = 2^(n^2).

Property	Condition
Reflexive	(a, a) in R for every a
Symmetric	(a, b) in R means (b, a) in R
Antisymmetric	(a, b) and (b, a) in R means $a = b$
Transitive	(a, b) and (b, c) in R means (a, c) in R

- **Equivalence relation** = reflexive + symmetric + transitive. It splits the set into **equivalence classes** (partitions).
- **Partial order (POSET)** = reflexive + antisymmetric + transitive. Example: $\subseteq$ on sets, divides ($|$) on positive integers.

FORMULA On n elements: reflexive relations = $2^{(n^2 - n)}$. Symmetric relations = $2^{(n(n+1)/2)}$. Antisymmetric = $2^n \cdot 3^{((n^2 - n)/2)}$.

➤ Lattices

- **Hasse diagram** = POSET drawing without loops and transitive edges.
- **Lattice** = POSET where every pair has a least upper bound (join) and greatest lower bound (meet).
- Distributive and complemented lattice = **Boolean algebra**.

3. Propositional Logic

- **Proposition** = a statement that is either true or false.

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

- $p \to q$ is false only when p is true and q is false.
- $p \to q \equiv \neg p \vee q$. Contrapositive $\neg q \to \neg p$ is equivalent to $p \to q$.
- Converse $q \to p$ and inverse $\neg p \to \neg q$ are NOT equivalent to $p \to q$.
- **Tautology**: always true. **Contradiction**: always false. **Contingency**: neither.

REMEMBER De Morgan: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Rules of inference

Rule	Form
Modus ponens	p, $p \to q$, so q
Modus tollens	$\neg q$, $p \to q$, so $\neg p$
Hypothetical syllogism	$p \to q$, $q \to r$, so $p \to r$
Disjunctive syllogism	$p \lor q$, $\neg p$, so q

Predicate logic

- $\forall x\ P(x)$: for all x. $\exists x\ P(x)$: there exists x.
- $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$ and $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$.
- "Every student likes some subject": $\forall x\ (Student(x) \to \exists y\ Likes(x, y))$.

4. Combinatorics

- **Sum rule**: m ways or n ways (not both) = $m + n$. **Product rule**: m ways then n ways = $m \times n$.

FORMULA Permutation $nPr = n! / (n - r)!$. Combination $nCr = n! / (r!\,(n - r)!)$.

Situation	Count
Arrange n distinct items in a circle	$(n - 1)!$
Arrange n items with repeats $n1, n2, \ldots$	$n! / (n1!\ n2!\ \ldots)$
Choose r from n with repetition	$C(n + r - 1, r)$
Non-negative solutions of $x1 + \ldots + xk = n$	$C(n + k - 1, k - 1)$
Derangements of n items	$D(n) = n!\,(1 - 1/1! + 1/2! - \ldots)$

EXAMPLE Arrangements of "BANANA" = $6! / (3!\ 2!) = 720 / 12 = 60$.

EXAMPLE Non-negative solutions of $x + y + z = 5 = C(7, 2) = 21$.

- Derangements: $D(1) = 0$, $D(2) = 1$, $D(3) = 2$, $D(4) = 9$, $D(5) = 44$.
- **Pigeonhole principle**: if $n + 1$ objects go into n boxes, at least one box has 2 or more. General: at least $\lceil N/k \rceil$ in some box.

Recurrence relations

- Linear homogeneous: $a(n) = c1 \cdot a(n-1) + c2 \cdot a(n-2)$. Solve with characteristic equation $r^2 - c1 \cdot r - c2 = 0$.

- Distinct roots r1, r2: $a(n) = A \cdot r1^n + B \cdot r2^n$. Repeated root r: $a(n) = (A + Bn) \cdot r^n$.

EXAMPLE $a(n) = 5a(n-1) - 6a(n-2)$, $a0 = 1$, $a1 = 4$. Roots 2 and 3. $a(n) = A \cdot 2^n + B \cdot 3^n$. $A + B = 1$, $2A + 3B = 4$, so $B = 2$, $A = -1$. $a(n) = 2 \cdot 3^n - 2^n$.

5. Group Theory

- **Group** $(G, *)$ needs: closure, associativity, identity, inverse.
- **Abelian group**: also commutative ($a * b = b * a$).
- Semigroup = closure + associative. Monoid = semigroup + identity.

Structure	Closure	Assoc.	Identity	Inverse	Commutative
Semigroup	Yes	Yes	-	-	-
Monoid	Yes	Yes	Yes	-	-
Group	Yes	Yes	Yes	Yes	-
Abelian group	Yes	Yes	Yes	Yes	Yes

- **Order of a group** = number of elements. **Order of an element** a = smallest n with $a^n = e$.
- **Lagrange's theorem**: order of a subgroup divides the order of the group.
- **Cyclic group**: generated by one element. Every cyclic group is abelian. Every group of prime order is cyclic.
- Example: $(\mathbb{Z}, +)$ is an abelian group. $(\mathbb{Z}, \times)$ is not a group (no inverse for 2).

6. Graph Theory

- Graph $G = (V, E)$. **Handshaking lemma**: sum of degrees = $2|E|$. So the number of odd-degree vertices is even.
- **Complete graph** K_n has $n(n-1)/2$ edges. **Tree** with n vertices has $n - 1$ edges.
- **Bipartite**: vertices split in two sets, edges only across. A graph is bipartite if and only if it has no odd cycle.

Concept	Condition
Euler circuit	Connected and every vertex has even degree
Euler path (not circuit)	Exactly 2 odd-degree vertices
Hamiltonian cycle	Visits every vertex once (no simple test, NP-complete)
Planar graph	Can be drawn with no edge crossing

FORMULA Euler's formula for connected planar graph: $V - E + F = 2$. For simple planar graphs with $V >= 3$: $E <= 3V - 6$.

- $K5$ and $K3,3$ are not planar (Kuratowski).
- **Chromatic number** $\chi(G)$ = minimum colours so adjacent vertices differ. Planar graphs: $\chi <= 4$. Bipartite (with an edge): $\chi = 2$. Kn: $\chi = n$.
- Number of labelled spanning trees of Kn = $n^{(n-2)}$ (Cayley). For $K4$: 16.
- **Matching**, **vertex cover**, **independent set**: in any graph, size of min vertex cover + size of max independent set = $|V|$.

7. Probability (GATE part)

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Conditional: $P(A \mid B) = P(A \cap B) / P(B)$.
- **Independent**: $P(A \cap B) = P(A) \cdot P(B)$.

FORMULA Bayes: $P(A \mid B) = P(B \mid A) \cdot P(A) / P(B)$.

Distribution	Mean	Variance
Binomial (n, p)	np	$np(1 - p)$
Poisson (λ)	λ	λ
Uniform (a, b)	$(a + b)/2$	$(b - a)^2 / 12$
Exponential (λ)	$1/\lambda$	$1/\lambda^2$

EXAMPLE Two fair dice. $P(\text{sum} = 7) = 6/36 = 1/6$.

8. Most Asked Questions (practice)

1. Prove that a relation is an equivalence relation and find its classes.
2. Draw the Hasse diagram of divisors of 24 under "divides". Is it a lattice?
3. Check if a given formula is a tautology using a truth table.

4. Use rules of inference to prove an argument is valid.
5. Translate English statements into predicate logic.
6. Solve a linear recurrence relation using the characteristic equation.
7. Count arrangements using permutations, combinations and the pigeonhole principle.
8. Prove that every cyclic group is abelian. State Lagrange's theorem.
9. Explain Euler and Hamiltonian graphs with examples. Verify Euler's formula.
10. Find the chromatic number of a given graph.

TIP GATE asks counting (relations, functions, combinations), logic translation, group properties and graph theory (degree, planarity, colouring). Learn every formula in the boxes.