Laplace and Fourier Transforms

B.Tech all branches · GATE Engineering Mathematics

Uncodemy Handwritten Notes · free for students

1. Laplace Transform: Definition

- For f(t) defined for t >= 0:

FORMULA L{f(t)} = F(s) = ∫ (0 to inf) e^(-st) f(t) dt.

- It changes a **differential equation in t** into an **algebraic equation in s**.
- Exists if f is piecewise continuous and of **exponential order** (abs f(t) <= M e^(at)).
- L is **linear**: L{a f + b g} = a F(s) + b G(s).

2. Standard Laplace Transforms

f(t)	F(s)	Valid for
1	1/s	s > 0
t	1/s^2	s > 0
t^n	n! / s^(n+1)	s > 0
e^(at)	1/(s - a)	s > a
sin at	a / (s^2 + a^2)	s > 0
cos at	s / (s^2 + a^2)	s > 0
sinh at	a / (s^2 - a^2)	s > abs(a)
cosh at	s / (s^2 - a^2)	s > abs(a)
t^(1/2)	sqrt(π) / (2 s^(3/2))	s > 0
t^(-1/2)	sqrt(π / s)	s > 0
u(t - a) (unit step)	e^(-as) / s	s > 0
δ(t - a) (impulse)	e^(-as)	all s

EXAMPLE $\mathcal{L}\{3t^2 + 2\sin 3t\} = 3(2/s^3) + 2(3/(s^2 + 9)) = 6/s^3 + 6/(s^2 + 9)$.

3. Properties of Laplace Transform

Property	Rule
First shifting	$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$
Second shifting	$\mathcal{L}\{f(t - a)\, u(t - a)\} = e^{-as} F(s)$
Change of scale	$\mathcal{L}\{f(at)\} = (1/a) F(s/a)$
Multiply by t	$\mathcal{L}\{t f(t)\} = -F'(s)$. In general $\mathcal{L}\{t^n f\} = (-1)^n d^n F/ds^n$
Divide by t	$\mathcal{L}\{f(t)/t\} = \int_s^\infty F(s)\, ds$
Derivative	$\mathcal{L}\{f'(t)\} = s F(s) - f(0)$
Second derivative	$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$
Integral	$\mathcal{L}\{\int_0^t f(u)\, du\} = F(s)/s$
Periodic (period T)	$\mathcal{L}\{f\} = [1/(1 - e^{-sT})] \int_0^T e^{-st} f(t)\, dt$

FORMULA Initial value: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s F(s)$. Final value: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s F(s)$ (if the limit exists).

➤ Solved examples

- $\mathcal{L}\{e^{2t} \sin 3t\}$: $\mathcal{L}\{\sin 3t\} = 3/(s^2 + 9)$. Shift $s \to s - 2$: $3 / ((s - 2)^2 + 9)$.
- $\mathcal{L}\{t \sin t\} = -d/ds\, [1/(s^2 + 1)] = 2s / (s^2 + 1)^2$.
- $\mathcal{L}\{(1 - e^{-t})/t\} = \int_s^\infty [1/s - 1/(s + 1)]\, ds = [\ln(s/(s + 1))]$ from s to $\infty = 0 - \ln(s/(s+1)) = \ln((s + 1)/s)$.

TIP Evaluating integrals with Laplace: $\int_0^\infty e^{-st} f(t)\, dt = F(s)$. Example: $\int_0^\infty t\, e^{-2t}\, dt = \mathcal{L}\{t\}$ at $s = 2 = 1/4$.

4. Inverse Laplace Transform

- $\mathcal{L}^{-1}\{F(s)\} = f(t)$. Main tools: tables, partial fractions, shifting, convolution.

F(s)	f(t)
1/(s - a)	e^(at)
1/((s - a)^2 + b^2)	(1/b) e^(at) sin bt
(s - a)/((s - a)^2 + b^2)	e^(at) cos bt
1/(s - a)^n	e^(at) t^(n-1)/(n-1)!
e^(-as) F(s)	f(t - a) u(t - a)

➤ Solved example (partial fractions)

- L^-1{1 / ((s - 1)(s - 2))}.
- 1/((s - 1)(s - 2)) = A/(s - 1) + B/(s - 2). A = 1/(1 - 2) = -1, B = 1/(2 - 1) = 1.
- f(t) = e^(2t) - e^t.

➤ Solved example (complete the square)

- L^-1{(s + 3)/(s^2 + 2s + 5)} = L^-1{((s + 1) + 2)/((s + 1)^2 + 4)}.
- = e^(-t) cos 2t + (2/2) e^(-t) sin 2t = e^(-t)(cos 2t + sin 2t).

➤ Convolution theorem

FORMULA L^-1{F(s) G(s)} = (f * g)(t) = ∫ (0 to t) f(u) g(t - u) du.

EXAMPLE L^-1{1/(s(s + 1))}: f = 1, g = e^(-t). ∫ (0 to t) e^(-(t-u)) du = 1 - e^(-t). (Partial fractions give the same.)

5. Solving ODEs with Laplace

- Steps: take L of both sides, put initial values, solve for Y(s), then take inverse.

➤ Solved example

- y'' + 3y' + 2y = 0, y(0) = 1, y'(0) = 0.
- [s^2 Y - s] + 3[sY - 1] + 2Y = 0 ⇒ Y(s^2 + 3s + 2) = s + 3.
- Y = (s + 3)/((s + 1)(s + 2)) = 2/(s + 1) - 1/(s + 2).
- y = 2e^(-t) - e^(-2t). Check: y(0) = 1, y'(0) = -2 + 2 = 0. Correct.

REMEMBER Laplace uses initial conditions directly, so there is no need to find CF and PI separately.

➤ Unit step and impulse

- $u(t-a) = 0$ for $t < a$, 1 for $t \geq a$. It switches a signal on at $t = a$.
- A pulse that is 1 between a and b: $u(t-a) - u(t-b)$, with transform $(e^{-as} - e^{-bs})/s$.
- $\delta(t-a)$ is a very short, very large spike of area 1. $\int f(t)\,\delta(t-a)\,dt = f(a)$.

EXAMPLE $L\{(t-2)^2\,u(t-2)\} = e^{-2s} \cdot 2/s^3$ (second shifting with $f(t) = t^2$).

➤ Solved example (first order with Laplace)

- $y' + 2y = 4$, $y(0) = 0$.
- $sY + 2Y = 4/s \Rightarrow Y = 4/(s(s+2)) = 2/s - 2/(s+2)$.
- $y = 2(1 - e^{-2t})$. Check: $y' = 4e^{-2t}$, $y' + 2y = 4e^{-2t} + 4 - 4e^{-2t} = 4$. Correct.
- Final value theorem: $\lim s\,Y = \lim 4/(s+2)$ at $s \to 0 = 2$. Matches $y(\infty) = 2$.

6. Fourier Series

- A periodic function $f(x)$ of period $2L$ (satisfying Dirichlet conditions) can be written as:

FORMULA $f(x) = a0/2 + \Sigma\,[an\cos(n\pi x/L) + bn\sin(n\pi x/L)]$, $n = 1$ to ∞.

FORMULA $a0 = (1/L)\int_{-L}^{L} f\,dx$, $an = (1/L)\int_{-L}^{L} f\cos(n\pi x/L)\,dx$, $bn = (1/L)\int_{-L}^{L} f\sin(n\pi x/L)\,dx$.

- For period 2π, put $L = \pi$.
- **Even function** ($f(-x) = f(x)$): $bn = 0$, only cosine terms.
- **Odd function** ($f(-x) = -f(x)$): $a0 = an = 0$, only sine terms.
- At a jump, the series gives the average of left and right limits.
- **Half-range series**: on $(0, L)$ only. Sine series (extend as odd) or cosine series (extend as even).

➤ Solved example

- $f(x) = x$ on $(-\pi, \pi)$. It is odd, so $a0 = an = 0$.

- $b_n = (2/\pi)\int_0^{\pi} x \sin nx\,dx = (2/\pi)[\pi(-1)^{n+1}/n] = 2(-1)^{n+1}/n$.
- $x = 2[\sin x - \sin 2x/2 + \sin 3x/3 - \dots]$.
- Put $x = \pi/2$: $\pi/2 = 2(1 - 1/3 + 1/5 - \dots) \Rightarrow 1 - 1/3 + 1/5 - \dots = \pi/4$.

FORMULA Parseval (period 2π): $(1/\pi)\int_{-\pi}^{\pi} f^2\,dx = a_0^2/2 + \sum(a_n^2 + b_n^2)$.

7. Fourier Transform

FORMULA $F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt$, and $f(t) = (1/2\pi)\int_{-\infty}^{\infty} F(\omega)\,e^{i\omega t}\,d\omega$.

- (Some books put $1/\sqrt{2\pi}$ in front of both. Use the convention of your syllabus.)
- **Fourier sine / cosine transforms** are used for functions on $(0, \infty)$.

Property	Rule
Linearity	$af + bg \to aF + bG$
Time shift	$f(t-a) \to e^{-i\omega a}F(\omega)$
Frequency shift	$e^{iat}f(t) \to F(\omega - a)$
Scaling	$f(at) \to (1/\lvert a\rvert)F(\omega/a)$
Derivative	$f'(t) \to i\omega F(\omega)$
Convolution	$f * g \to F(\omega)G(\omega)$

➤ Standard pairs (with the convention above)

$f(t)$	$F(\omega)$
$\delta(t)$	1
$e^{-a\lvert t\rvert}$, $a > 0$	$2a/(a^2 + \omega^2)$
$e^{-at}u(t)$, $a > 0$	$1/(a + i\omega)$
Rectangular pulse 1 for $\lvert t\rvert < a$	$2\sin(a\omega)/\omega$

EXAMPLE $e^{-t}u(t)$: $F = \int_0^{\infty} e^{-t}e^{-i\omega t}\,dt = 1/(1 + i\omega)$.

➤ Z-transform (short note)

- $Z\{x(n)\} = \sum x(n)z^{-n}$. $Z\{a^n u(n)\} = z/(z - a)$, for $\lvert z\rvert > \lvert a\rvert$. Used for difference equations.

8. Most Asked Questions (practice)

1. Find L{t^2 e^(-3t)} and L{e^(-t) cos 2t}.
2. Find L{(sin t)/t} and hence evaluate ∫ (0 to inf) (sin t)/t dt.
3. Find the Laplace transform of a square wave (periodic function).
4. Find L^-1{s / ((s^2 + 1)(s^2 + 4))} by partial fractions.
5. Using convolution, find L^-1{1 / (s^2 (s + 1))}.
6. Solve y'' + 4y = sin t, y(0) = 0, y'(0) = 0 using Laplace.
7. Find the Fourier series of f(x) = x^2 on (-π, π) and deduce Σ 1/n^2 = π^2/6.
8. Find the half-range cosine series of f(x) = x on (0, π).
9. Find the Fourier transform of e^(-a abs(x)).
10. State and prove the first shifting theorem.

TIP For GATE, the Laplace table, shifting theorems and final value theorem are must-know. For universities, Fourier series of x and x^2 and solving ODEs by Laplace come almost every year.