Linear Algebra and Matrices

B.Tech all branches · GATE Engineering Mathematics

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1. Matrices: Basics

- **Matrix** = a rectangular array of numbers with m rows and n columns (order m x n).
- **Square matrix**: m = n. **Diagonal**: only diagonal entries non-zero. **Identity I**: diagonal entries all 1.
- **Transpose** A^T: rows become columns. (AB)^T = B^T A^T.
- Matrix product AB is defined only if columns of A = rows of B. In general AB ≠ BA.

➤ Special matrices

Matrix	Condition	Note
Symmetric	A^T = A	Eigenvalues real
Skew-symmetric	A^T = -A	Diagonal entries 0, eigenvalues 0 or pure imaginary
Orthogonal	A^T A = I	A^-1 = A^T, det = ±1
Hermitian	(conj A)^T = A	Eigenvalues real
Unitary	(conj A)^T A = I	abs of each eigenvalue = 1
Idempotent	A^2 = A	Eigenvalues 0 or 1
Nilpotent	A^k = 0 for some k	All eigenvalues 0
Involutory	A^2 = I	Eigenvalues ±1

REMEMBER Every square matrix = symmetric part + skew part: A = (A + A^T)/2 + (A - A^T)/2.

2. Determinants and Inverse

- 2 x 2: det [a b; c d] = ad - bc.

- det(AB) = det A . det B. det(A^T) = det A. det(kA) = k^n det A for an n x n matrix.
- Swapping two rows changes the sign. Two equal rows ⇒ det = 0.
- **Singular** matrix: det = 0 (no inverse).

FORMULA A^-1 = adj(A) / det(A). For 2 x 2: [a b; c d]^-1 = (1/(ad - bc)) [d -b; -c a].

- (AB)^-1 = B^-1 A^-1. det(A^-1) = 1 / det A.
- det(adj A) = (det A)^(n-1) for an n x n matrix.

EXAMPLE A = [2 1; 5 3]. det = 6 - 5 = 1. A^-1 = [3 -1; -5 2]. Check: A A^-1 = [6-5, -2+2; 15-15, -5+6] = I.

3. Rank of a Matrix

- **Rank** = number of non-zero rows in **echelon form** = largest order of a non-zero minor.
- = number of linearly independent rows (= columns).
- rank(A) <= min(m, n). rank(AB) <= min(rank A, rank B).
- For an n x n matrix: rank = n <⇒ det ≠ 0 <⇒ A is invertible.

➤ Solved example

- A = [1 2 3; 2 4 6; 1 1 1].
- R2 → R2 - 2R1 gives [0 0 0]. R3 → R3 - R1 gives [0 -1 -2].
- Non-zero rows: [1 2 3] and [0 -1 -2] ⇒ rank = 2.

TIP Rank of a non-zero column times a non-zero row (like u v^T) is always 1.

4. System of Linear Equations

- Write as **AX = B**. Augmented matrix [A | B].
- Homogeneous AX = 0 always has the trivial solution X = 0.

Consistency (Rouche-Capelli test), n unknowns

Condition	Result
rank A ≠ rank [A:B]	No solution (inconsistent)
rank A = rank [A:B] = n	Unique solution
rank A = rank [A:B] = r < n	Infinite solutions, (n - r) free variables

- Here [A:B] means the augmented matrix [A | B].
- Homogeneous system has non-trivial solutions <=> rank < n <=> det A = 0 (square case).
- Number of independent solutions of AX = 0 = **nullity** = n - rank.

FORMULA Rank-nullity theorem: rank(A) + nullity(A) = n (number of columns).

Methods

- **Gauss elimination**: make upper triangular, then back substitution.
- **Gauss-Jordan**: make it diagonal (reduced row echelon).
- **Cramer's rule**: x_i = det(A_i) / det(A), only when det A ≠ 0.
- **LU decomposition**: A = LU, solve LY = B then UX = Y.

Solved example

- x + y = 3, 2x - y = 0.
- Add: 3x = 3 ⇒ x = 1. Then y = 2. Solution (1, 2).
- Cramer: det A = (1)(-1) - (1)(2) = -3. det A_x = (3)(-1) - (1)(0) = -3 ⇒ x = 1. Same answer.

5. Vector Spaces

- **Vector space**: a set closed under addition and scalar multiplication (with the usual rules).
- **Linearly independent**: c1 v1 + ... + cn vn = 0 only when all c = 0.
- **Basis**: an independent set that spans the space. **Dimension** = number of vectors in a basis.
- dim R^n = n. Any n + 1 vectors in R^n are dependent.
- Vectors in R^n are independent <=> matrix made from them has rank = number of vectors.

Four subspaces of an m x n matrix of rank r

Subspace	Dimension
Column space	r
Row space	r
Null space	n - r
Left null space	m - r

6. Linear Transformations

- T: V → W is linear if T(au + bv) = aT(u) + bT(v).
- Every linear map between finite spaces is a matrix multiplication.
- **Kernel** (null space) and **range** (image). dim ker T + dim range T = dim V.
- Rotation by θ in 2D: [cos θ -sin θ; sin θ cos θ] (orthogonal, det = 1).

7. Eigenvalues and Eigenvectors

- A X = λ X with X ≠ 0. λ = **eigenvalue**, X = **eigenvector**.
- **Characteristic equation**: det(A - λI) = 0.

Properties (very important for GATE)

- Sum of eigenvalues = **trace** (sum of diagonal). Product of eigenvalues = **det A**.
- Eigenvalues of a triangular / diagonal matrix = its diagonal entries.
- A and A^T have the same eigenvalues.
- If λ is an eigenvalue of A: A^k has λ^k, A^-1 has 1/λ, kA has kλ, A + kI has λ + k.
- det A = 0 <⇒ at least one eigenvalue is 0.
- Eigenvectors of distinct eigenvalues are independent. For a symmetric matrix they are orthogonal.

FORMULA 2 x 2 matrix: λ^2 - (trace) λ + det = 0.

Solved example

- A = [4 1; 2 3]. Trace = 7, det = 12 - 2 = 10.
- λ^2 - 7λ + 10 = 0 ⇒ (λ - 5)(λ - 2) = 0 ⇒ λ = 5, 2.
- λ = 5: (A - 5I)X = [-1 1; 2 -2]X = 0 ⇒ -x + y = 0 ⇒ X = (1, 1).

- $\lambda = 2$: $(A - 2I)X = [2\ 1;\ 2\ 1]X = 0 \Rightarrow 2x + y = 0 \Rightarrow X = (1, -2)$.
- Check: $A(1, 1) = (5, 5) = 5(1, 1)$. $A(1, -2) = (2, -4) = 2(1, -2)$. Correct.

➤ Cayley–Hamilton theorem

- Every square matrix satisfies its own characteristic equation.
- For the A above: $A^2 - 7A + 10I = 0$. So $A^{-1} = (7I - A)/10$.

> **EXAMPLE** $A^{-1} = (1/10)[7-4, -1; -2, 7-3] = (1/10)[3\ -1; -2\ 4]$. Check with 2 x 2 formula: $(1/10)[3\ -1; -2\ 4]$. Same.

8. Diagonalization and Quadratic Forms

- A is **diagonalizable** if it has n independent eigenvectors. Then $P^{-1} A P = D$ (D = diagonal of eigenvalues, P = eigenvectors as columns).
- Distinct eigenvalues ⇒ always diagonalizable. Real symmetric matrix ⇒ always (orthogonally) diagonalizable.
- Use: $A^k = P D^k P^{-1}$ (easy powers).
- **Algebraic multiplicity** (AM) = times λ repeats as a root. **Geometric multiplicity** (GM) = number of independent eigenvectors for λ. $GM <= AM$. Diagonalizable <⇒ $GM = AM$ for every λ.

➤ Quadratic forms

- $Q = X^T A X$ with A symmetric. Nature from eigenvalues of A:

Eigenvalues	Nature
All > 0	Positive definite
All >= 0, some = 0	Positive semi-definite
All < 0	Negative definite
Some > 0, some < 0	Indefinite

> **TIP** Positive definite check without eigenvalues: all leading principal minors > 0.

9. Most Asked Questions (practice)

1. Find the rank of a 3 x 4 matrix by reducing to echelon form.
2. Test consistency and solve $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 9z = 36$.
3. For what value of k does a homogeneous system have non-trivial solutions?

4. Find eigenvalues and eigenvectors of [2 0 1; 0 2 0; 1 0 2].
5. Verify Cayley–Hamilton theorem for a 2 x 2 matrix and find A^-1.
6. Diagonalize a symmetric matrix and find A^4.
7. Prove that eigenvalues of a real symmetric matrix are real.
8. Check if vectors (1, 2, 3), (2, 3, 4), (3, 4, 5) are independent.
9. Reduce a quadratic form to canonical form and state its nature.
10. State the rank–nullity theorem and verify it for a given matrix.

TIP For GATE, master eigenvalue properties (trace, det, powers), rank and consistency. Most questions can be solved in one minute with these shortcuts, without full calculation.