Mechanical Vibrations

Short notes for B.Tech Mechanical · GATE ME

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1. Basics

- **Vibration** = repeated back-and-forth motion of a body about its mean (equilibrium) position.
- It needs: a mass (stores KE), a spring (stores PE) and usually a damper (removes energy).
- Good uses: vibrating screens, conveyors, musical instruments. Bad effects: noise, wear, fatigue failure, discomfort.

➤ Terms

Term	Meaning
Time period T	Time for one cycle
Frequency f	Cycles per second (Hz), f = 1/T
Amplitude	Max displacement from mean position
Natural frequency	Frequency of free vibration (no outside force)
Resonance	Forcing frequency = natural frequency; very large amplitude
Degrees of freedom (DOF)	Number of independent coordinates needed to describe motion

➤ Types of vibration

Basis	Types
Outside force	Free, forced
Energy loss	Undamped, damped
Direction	Longitudinal, transverse, torsional
Nature	Deterministic, random
System	Linear, non-linear

- **SHM**: $x = X \sin(\omega t)$. Velocity max = $X\omega$, acceleration max = $X\omega^2$. Acceleration is proportional to displacement and acts towards the mean position.

2. Undamped Free Vibration (Single DOF)

- Equation of motion: $m\ddot{x} + kx = 0$.

FORMULA Natural frequency $\omega_n = \sqrt{k/m}$ rad/s, $f_n = (1/2\pi)\sqrt{k/m}$ Hz = $(1/2\pi)\sqrt{g/\delta_{st}}$.

- δ_{st} = static deflection = mg/k. $f_n \approx 0.4985/\sqrt{\delta_{st}}$ Hz (δ_{st} in metres).

EXAMPLE A machine causes a static deflection of 2 mm on its springs. $f_n = (1/2\pi)\sqrt{9.81/0.002} = 70.04/6.283 =$ 11.15 Hz.

Methods to find natural frequency

1. **Equilibrium (Newton) method**: $\Sigma F = m\ddot{x}$.
2. **Energy method**: total energy KE + PE = constant, so $d(KE + PE)/dt = 0$.
3. **Rayleigh's method**: max KE = max PE. It gives a value at or above the true fundamental frequency (upper bound).

Equivalent springs

FORMULA Springs in series: $1/k = 1/k1 + 1/k2$. Springs in parallel: $k = k1 + k2$.

System	Stiffness k
Cantilever, load at free end	$3EI/L^3$
Simply supported, load at centre	$48EI/L^3$
Fixed–fixed, load at centre	$192EI/L^3$
Bar in axial load	AE/L
Shaft in torsion	GJ/L (N·m/rad)

Other single DOF systems

- Simple pendulum: $\omega_n = \sqrt{g/L}$.
- Compound pendulum: $\omega_n = \sqrt{g\,h/(k_G^2 + h^2)}$, h = distance from pivot to CG.
- Spring mass effect: if the spring mass m_s is not small, use $m + m_s/3$.

3. Damped Free Vibration

- **Viscous damping**: damping force = c x' (c = damping coefficient, N·s/m).
- Equation: m x'' + c x' + k x = 0.

FORMULA Critical damping cc = 2 sqrt(k m) = 2 m ωn. Damping ratio ζ = c / cc.

ζ	Type	Motion
ζ < 1	Under-damped	Oscillates with decaying amplitude
ζ = 1	Critically damped	Returns fastest without oscillating
ζ > 1	Over-damped	Returns slowly without oscillating

FORMULA Damped natural frequency ωd = ωn sqrt(1 − ζ^2). Logarithmic decrement δ = ln(x0/x1) = 2πζ / sqrt(1 − ζ^2) = (1/n) ln(x0/xn).

- Critical damping is used in gun recoil mechanisms and meters (needle settles fast without swinging).

EXAMPLE m = 10 kg, k = 4000 N/m, c = 80 N·s/m. Find ωn, ζ, ωd and the ratio of successive amplitudes.

- ωn = sqrt(4000/10) = 20 rad/s (fn = 3.18 Hz).
- cc = 2 sqrt(4000 x 10) = 400 N·s/m. ζ = 80/400 = 0.2.
- ωd = 20 sqrt(1 − 0.04) = 20 x 0.980 = 19.6 rad/s.
- δ = 2π x 0.2 / 0.980 = 1.283. Amplitude ratio x0/x1 = e^1.283 = 3.61.

➤ Coulomb (dry friction) damping

- Friction force is constant. Amplitude drops linearly, by 4F/k every cycle (viscous damping gives an exponential drop).
- Frequency stays equal to ωn. Motion stops when the spring force can no longer beat friction.

4. Forced Vibration

- Harmonic force F0 sin(ωt) on a damped spring–mass system.
- Frequency ratio r = ω / ωn. Static deflection Xst = F0 / k.

FORMULA Magnification factor MF = X / Xst = 1 / sqrt((1 − r^2)^2 + (2ζr)^2). Phase angle tanϕ = 2ζr / (1 − r^2).

- At resonance (r = 1): MF = 1/(2ζ), phase = 90°. Undamped system: amplitude tends to infinity.
- Peak amplitude occurs at r = sqrt(1 − 2ζ^2), a little below 1.
- r << 1: spring controls. r near 1: damping controls. r >> 1: mass controls (amplitude small).

➤ Rotating unbalance

FORMULA Amplitude X = (m0 e / M) x r^2 / sqrt((1 − r^2)^2 + (2ζr)^2) (m0 = unbalanced mass at eccentricity e, M = total mass).

➤ Base (support) excitation

- Displacement transmissibility X/Y has the same formula as force transmissibility below.

5. Vibration Isolation and Transmissibility

- **Transmissibility** TR = force passed to the foundation / applied force.

FORMULA TR = sqrt(1 + (2ζr)^2) / sqrt((1 − r^2)^2 + (2ζr)^2). Undamped: TR = 1 / |1 − r^2|.

- TR < 1 (isolation) only when r > sqrt(2). So use soft springs (low ωn) under machines.
- For r > sqrt(2), more damping actually increases TR; but some damping is needed to pass safely through resonance at start-up.
- All curves of TR (for any ζ) pass through TR = 1 at r = sqrt(2).

EXAMPLE A 100 kg machine on springs of total k = 10^5 N/m runs at 600 rpm (no damping). Find TR.

- ωn = sqrt(10^5/100) = 31.62 rad/s. ω = 2π x 600/60 = 62.83 rad/s.
- r = 1.987, r^2 = 3.948.

- TR = 1 / (3.948 − 1) = 0.339, so 33.9% of the force reaches the floor.

6. Torsional Vibration

FORMULA Single rotor on a shaft: ωn = sqrt(q / I), q = G J / L (torsional stiffness).

- **Two-rotor system**: rotors spin in opposite directions about a **node** (point of zero twist). Node position: IA LA = IB LB.
- Frequency of a two-rotor system: fn = (1/2π) sqrt(G J / (IA LA)).
- **Three-rotor system**: two nodes, two non-zero natural frequencies.
- **Equivalent shaft**: replace a stepped shaft by one of uniform diameter d with L = L1 + L2 (d/d2)^4 + ...
- Geared systems: refer inertia and stiffness to one shaft using (gear ratio)^2.

7. Whirling of Shafts (Critical Speed)

- **Critical (whirling) speed** = speed at which a rotating shaft vibrates violently; it equals the natural frequency of transverse vibration.
- Causes: eccentricity of rotor mass, shaft weight, unbalance.

FORMULA Critical speed ωc = sqrt(k/m) = sqrt(g/δ). Dunkerley's method: 1/ωc^2 = 1/ω1^2 + 1/ω2^2 + ... + 1/ωs^2.

- Dunkerley gives a lower bound (value at or below the true critical speed). Rayleigh gives an upper bound.
- Above the critical speed, the mass centre moves towards the shaft axis (self-centring); turbines often run above critical speed.

8. Two DOF Systems and Measurement

- A 2-DOF system has two natural frequencies and two **mode shapes** (principal modes).
- Examples: two masses with springs, car body bounce and pitch, double pendulum.

- **Dynamic (tuned) vibration absorber**: attach a small spring–mass (k2, m2) with sqrt(k2/m2) = forcing ω. The main mass then stays still. Drawback: two new resonance peaks appear near the old one.
- **Semi-definite (degenerate) system**: one natural frequency is zero (rigid body motion), e.g. two rotors on a free shaft.

➤ Vibration measuring instruments

Instrument	Natural frequency	Frequency ratio	Measures
Vibrometer (seismometer)	Low	r large (> 3)	Displacement
Accelerometer	High	r small	Acceleration
Velometer	Low	r large	Velocity

- **FFT analyser**: converts time signal to frequency spectrum; used for machine condition monitoring.

9. Most Asked Questions (practice)

1. Define natural frequency, resonance and degrees of freedom. Classify vibrations.
2. Find natural frequency by equilibrium, energy and Rayleigh's methods.
3. Find natural frequency of a compound pendulum.
4. Derive the equation of motion of a damped system. Explain under, critical and over damping.
5. Define logarithmic decrement and derive its relation with ζ.
6. Compare viscous and Coulomb damping.
7. Derive magnification factor and phase angle for forced vibration. Sketch the curves.
8. Explain transmissibility and vibration isolation. Why is r > sqrt(2) needed?
9. Find natural frequency and node position of a two-rotor torsional system.
10. Explain critical speed of shafts and Dunkerley's method.

TIP In GATE ME, almost every year has a question on ωn from equivalent stiffness, damping ratio / log decrement, magnification factor or transmissibility. Learn ζ = c/(2 sqrt(km)) and r > sqrt(2) by heart.