Signals and Systems

Short notes for B.Tech ECE · EEE · GATE EC

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I. Signals: Basics and Classification

- **Signal** = a quantity that changes with time (or space) and carries information. Example: speech, ECG, temperature.
- **Continuous-time (CT)** x(t): defined for all t. **Discrete-time (DT)** x[n]: defined only at integer n.

➤ Classification

Type	Meaning	Example
Periodic	x(t) = x(t + T) for all t	sin(ωt)
Aperiodic	No such T	e^(−t) u(t)
Even	x(−t) = x(t)	cos t
Odd	x(−t) = −x(t)	sin t
Energy signal	0 < E < ∞, P = 0	Finite pulse, e^(−at) u(t)
Power signal	0 < P < ∞, E = ∞	Periodic signals, u(t)
Deterministic	Exact formula known	cos(2πt)
Random	Only statistics known	Noise

FORMULA Energy E = integral of |x(t)|^2 dt over all t. Power P = lim T→∞ (1/2T) integral from −T to T of |x(t)|^2 dt.

- Even part: xe(t) = [x(t) + x(−t)] / 2. Odd part: xo(t) = [x(t) − x(−t)] / 2.
- A signal can be neither energy nor power (example: t u(t), e^t).

➤ Periodicity rules

- Sum of two periodic CT signals is periodic only if T1 / T2 is a rational number. Then T = LCM(T1, T2).
- DT signal cos(ω0 n) is periodic only if ω0 / 2π = k / N is rational. Then the period is N (smallest integer).

EXAMPLE $x(t) = \cos(4t) + \sin(6t)$. $T1 = 2\pi/4 = \pi/2$, $T2 = 2\pi/6 = \pi/3$. $T1/T2 = 3/2$ (rational). $T = \text{LCM}(\pi/2, \pi/3) = \pi$.

➤ Standard signals

Signal	Definition	Key property
Unit step $u(t)$	1 for $t >= 0$, 0 for $t < 0$	$du/dt = \delta(t)$
Unit impulse $\delta(t)$	Zero except $t = 0$, area = 1	Sifting: integral $x(t)\ \delta(t - t0)\ dt = x(t0)$
Ramp $r(t)$	$t\ u(t)$	$dr/dt = u(t)$
Signum $\text{sgn}(t)$	+1 for $t > 0$, −1 for $t < 0$	$\text{sgn}(t) = 2u(t) - 1$
Sinc	$\text{sinc}(t) = \sin(\pi t) / (\pi t)$	$\text{sinc}(0) = 1$

- $\delta(at) = (1/|a|)\ \delta(t)$. $x(t)\ \delta(t) = x(0)\ \delta(t)$.

➤ Operations on signals

- Time shift $x(t - t0)$: delay if $t0 > 0$. Time reversal $x(-t)$. Time scaling $x(at)$: compress if $|a| > 1$.
- For $x(at + b)$, first shift by b, then scale by a (or write $x(a(t + b/a))$).

2. Systems and Their Properties

Property	Test	Example that fails
Linear	Additivity + homogeneity (superposition)	$y = x^2$, $y = x + 2$
Time-invariant	Delay in input → same delay in output	$y(t) = t\ x(t)$, $y(t) = x(2t)$
Causal	Output depends only on present and past input	$y(t) = x(t + 1)$
Memoryless (static)	Output depends only on present input	$y[n] = x[n - 1]$
Stable (BIBO)	Bounded input gives bounded output	$y(t) = \text{integral of } x(\tau)\ d\tau$
Invertible	Input can be recovered from output	$y = x^2$

REMEMBER Any system with a constant added ($y = x + 2$) is non-linear, because zero input does not give zero output.

➤ LTI system and impulse response

- An LTI system is fully described by its **impulse response** $h(t)$ (or $h[n]$).
- Causal LTI: $h(t) = 0$ for $t < 0$. Stable LTI: integral of $|h(t)|\,dt < \infty$ (DT: sum of $|h[n]| < \infty$).

FORMULA CT convolution: $y(t) = x(t) * h(t) = \int x(\tau)\, h(t - \tau)\, d\tau$. DT: $y[n] = \sum_k x[k]\, h[n - k]$.

- Properties: commutative, associative, distributive. $x(t) * \delta(t - t0) = x(t - t0)$.
- Length of DT convolution: if x has L samples and h has M samples, y has $L + M - 1$ samples.
- Cascade of LTI systems: $h = h1 * h2$. Parallel: $h = h1 + h2$.

➤ Solved example (DT convolution)

- $x[n] = \{1, 2, 1\}$, $h[n] = \{1, 1\}$ (both start at $n = 0$).
  - $y[0] = 1 \times 1 = 1$
  - $y[1] = 2 \times 1 + 1 \times 1 = 3$
  - $y[2] = 1 \times 1 + 2 \times 1 = 3$
  - $y[3] = 1 \times 1 = 1$
  - $y[n] = \{1, 3, 3, 1\}$, length $= 3 + 2 - 1 = 4$ (correct)

3. Fourier Series

- Any periodic signal (satisfying Dirichlet conditions) can be written as a sum of harmonics of $\omega0 = 2\pi / T$.

FORMULA Exponential FS: $x(t) = \sum Cn\, e^{(jn\omega0t)}$, $Cn = (1/T) \int$ over one period of $x(t)\, e^{(-jn\omega0t)}\, dt$.

FORMULA Trigonometric FS: $x(t) = a0 + \sum [an \cos(n\omega0t) + bn \sin(n\omega0t)]$, $a0$ = average value.

➤ Symmetry shortcuts

Symmetry	Result
Even x(t)	bn = 0 (only cosine terms)
Odd x(t)	a0 = 0, an = 0 (only sine terms)
Half-wave symmetry x(t) = −x(t − T/2)	Only odd harmonics

- **Dirichlet conditions**: absolutely integrable over a period, finite maxima/minima, finite discontinuities.
- **Gibbs phenomenon**: about 9% overshoot near a jump, does not reduce as more terms are added.

FORMULA Parseval (FS): average power P = sum of |Cn|^2 over all n.

4. Fourier Transform

FORMULA X(ω) = integral of x(t) e^(−jωt) dt. Inverse: x(t) = (1/2π) integral of X(ω) e^(jωt) dω.

➤ Important pairs

x(t)	X(ω)
δ(t)	1
1	2π δ(ω)
e^(−at) u(t), a > 0	1 / (a + jω)
e^(−a abs(t))	2a / (a^2 + ω^2)
Rectangular pulse, height 1, width τ	τ sinc(ωτ / 2π) = τ sin(ωτ/2) / (ωτ/2)
cos(ω0 t)	π [δ(ω − ω0) + δ(ω + ω0)]
sgn(t)	2 / (jω)
u(t)	π δ(ω) + 1 / (jω)

Properties

Property	Time domain	Frequency domain
Time shift	$x(t - t0)$	$e^{(-j\omega t0)}\ X(\omega)$
Frequency shift	$e^{(j\omega 0 t)}\ x(t)$	$X(\omega - \omega 0)$
Scaling	$x(at)$	$(1 / \lvert a\rvert)\ X(\omega/a)$
Differentiation	dx/dt	$j\omega\ X(\omega)$
Convolution	$x(t) * h(t)$	$X(\omega)\ H(\omega)$
Multiplication	$x(t)\ h(t)$	$(1/2\pi)\ X(\omega) * H(\omega)$
Duality	$X(t)$	$2\pi\ x(-\omega)$

FORMULA Parseval: $E = \int |x(t)|^2\ dt = (1/2\pi) \int |X(\omega)|^2\ d\omega$.

EXAMPLE Energy of $e^{(-at)}\ u(t) = \int_0^\infty e^{(-2at)}\ dt = 1/(2a)$. For $a = 2$, $E = 0.25$ J.

- Distortionless system: $H(\omega) = K\ e^{(-j\omega td)}$ (constant gain, linear phase). Group delay $= -d\theta/d\omega$.

5. Laplace Transform

FORMULA $X(s) = \int_{-\infty}^{\infty} x(t)\ e^{(-st)}\ dt$, $s = \sigma + j\omega$. Unilateral form starts from $0-$.

Common pairs (with ROC)

x(t)	X(s)	ROC
$\delta(t)$	1	All s
$u(t)$	$1/s$	$Re(s) > 0$
$e^{(-at)}\ u(t)$	$1/(s + a)$	$Re(s) > -a$
$-e^{(-at)}\ u(-t)$	$1/(s + a)$	$Re(s) < -a$
$t^n\ u(t)$	$n!\ /\ s^{(n+1)}$	$Re(s) > 0$
$\sin(\omega 0 t)\ u(t)$	$\omega 0\ /\ (s^2 + \omega 0^2)$	$Re(s) > 0$
$\cos(\omega 0 t)\ u(t)$	$s\ /\ (s^2 + \omega 0^2)$	$Re(s) > 0$

- **ROC rules**: no poles inside ROC. Right-sided signal → ROC to the right of the rightmost pole. Left-sided → left of the leftmost pole.

- Causal and stable LTI: all poles in the left half plane, ROC includes the jω axis.
- Initial value: x(0+) = lim s→∞ s X(s). Final value: x(∞) = lim s→0 s X(s).

EXAM POINT Same X(s) can belong to different signals. Always mention the ROC. Example above: 1/(s + a) is both a right-sided and a left-sided signal.

6. Z-Transform

FORMULA X(z) = sum over n of x[n] z^(−n). Relation to DTFT: put z = e^(jω).

x[n]	X(z)	ROC
δ[n]	1	All z
u[n]	1 / (1 − z^−1)	abs(z) > 1
a^n u[n]	1 / (1 − a z^−1)	abs(z) > abs(a)
−a^n u[−n − 1]	1 / (1 − a z^−1)	abs(z) < abs(a)
n a^n u[n]	a z^−1 / (1 − a z^−1)^2	abs(z) > abs(a)

- Time shift: x[n − k] → z^(−k) X(z). Convolution → product.
- Causal and stable DT system: all poles inside the unit circle, ROC includes |z| = 1.
- Inverse Z-transform methods: partial fractions, power series (long division), residues.

➤ Solved example (system function)

- y[n] = 0.5 y[n − 1] + x[n]. Taking Z-transform: Y(z) (1 − 0.5 z^−1) = X(z).
  - H(z) = 1 / (1 − 0.5 z^−1), pole at z = 0.5
  - Causal → h[n] = (0.5)^n u[n]
  - Pole inside unit circle → stable. Sum of |h[n]| = 1 / (1 − 0.5) = 2 (finite)

7. Sampling

- **Sampling theorem**: a band-limited signal with highest frequency fm can be recovered from samples if fs >= 2 fm.
- **Nyquist rate** = 2 fm. **Nyquist interval** = 1 / (2 fm).
- If fs < 2 fm, spectra overlap → aliasing. Use an anti-aliasing low-pass filter before sampling.
- Ideal reconstruction: low-pass filter with cutoff between fm and fs − fm.

EXAMPLE $x(t) = \cos(100\pi t) + \cos(300\pi t)$. Frequencies 50 Hz and 150 Hz. fm = 150 Hz, Nyquist rate = 300 Hz.

FORMULA Nyquist rate of $x^2(t)$ = 2 x (2 fm) (squaring doubles the bandwidth). For x1(t) x2(t), bandwidth = fm1 + fm2.

➤ Types of sampling

Type	Description
Ideal (impulse)	Multiply by impulse train
Natural	Multiply by pulse train, top follows the signal
Flat-top	Sample and hold, causes aperture effect (fix with equalizer)

8. Comparison of Transforms

Transform	Signal	Domain	Used for
Fourier series	CT periodic	Discrete frequencies	Harmonic analysis
Fourier transform	CT aperiodic	Continuous ω	Spectrum, filtering
Laplace	CT, even unstable	s-plane	Circuits, control, stability
DTFT	DT	Continuous ω, period 2π	DT spectrum
Z-transform	DT	z-plane	Difference equations, digital filters

9. Most Asked Questions (practice)

1. Check linearity, time invariance, causality and stability of $y(t) = t\,x(t)$ and $y[n] = x[2n]$.
2. Find the fundamental period of a sum of sinusoids (CT and DT).
3. Classify energy and power signals and find E or P.
4. Compute the convolution of two finite DT sequences and two CT signals.
5. Find the Fourier series of a square wave and a sawtooth wave.
6. State and prove the time-shift, convolution and Parseval properties of the Fourier transform.

7. Find the Laplace transform with ROC; explain causality and stability from the ROC.
8. Find the inverse Z-transform by partial fractions for all possible ROCs.
9. State the sampling theorem and explain aliasing with spectra.
10. Find the impulse response of a system from its difference equation.

TIP GATE asks quick questions on system properties, periodicity, ROC, convolution length and Nyquist rate. Learn the transform tables by heart.