Structural Analysis

Short notes for B.Tech Civil · GATE CE

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1. Basics and Determinacy

- **Statically determinate** structure: all reactions and internal forces can be found from equilibrium equations alone.
- **Statically indeterminate**: unknowns are more than equilibrium equations. Extra unknowns = **degree of static indeterminacy (Ds)**.
- **Kinematic indeterminacy (Dk)** = number of unknown joint displacements (degrees of freedom).
- Equations of equilibrium in a plane: $\Sigma Fx = 0$, $\Sigma Fy = 0$, $\Sigma M = 0$ (3 equations).

➤ Support reactions

Support	Reactions	Freedoms
Roller	1	2 (rotation + one translation)
Hinge (pin)	2	1 (rotation)
Fixed	3	0
Guided (slider)	2	1 (translation)

FORMULA Plane truss: $Ds = m + r - 2j$. Plane rigid frame: $Ds = 3m + r - 3j -$ (releases). Space truss: $Ds = m + r - 3j$.

FORMULA Plane frame kinematic indeterminacy: $Dk = 3j - r$ (with axial deformation). Neglect axial deformation: subtract the number of members that do not change length.

- If $Ds < 0$ the structure is a mechanism (unstable). $Ds = 0$ means determinate.
- A structure can also be **geometrically unstable** even when $Ds \geq 0$ (all reactions parallel or concurrent).

EXAMPLE Plane truss with m = 10 members, r = 4 reactions, j = 6 joints. Ds = 10 + 4 - 2 x 6 = 2. So the truss is indeterminate to the 2nd degree.

➤ Analysis of plane trusses

- Assumptions: pin joints, loads only at joints, members carry only axial force.
- **Method of joints**: equilibrium of each joint (2 equations).
- **Method of sections**: cut at most 3 unknown members, take moments.
- Zero-force members: at an unloaded joint with two non-collinear members, both are zero. At an unloaded joint with three members, two collinear, the third is zero.

2. Energy Methods and Deflection

- **Strain energy** in bending: $U = \int M^2\,dx / (2EI)$. In axial: $U = \Sigma P^2 L / (2AE)$.
- **Castigliano's 1st theorem**: deflection under a load = $\partial U / \partial P$.
- **Unit load method** (virtual work): $\delta = \int M m\,dx / (EI)$. Trusses: $\delta = \Sigma P k L / (AE)$, where k = force due to unit load.
- **Maxwell's reciprocal theorem**: deflection at A due to unit load at B = deflection at B due to unit load at A ($\delta ab = \delta ba$).
- **Betti's theorem**: general form of Maxwell for two sets of loads.

TIP For deflection at a point where no load acts, apply a dummy load Q there, use Castigliano, then set Q = 0.

3. Indeterminate Beams: Force Methods

- **Consistent deformation** (flexibility method): remove redundant, find deflection, apply redundant to restore compatibility.
- **Clapeyron's three moment theorem** for continuous beams (uniform EI, supports at same level):

FORMULA Three moment: $MA\,L1 + 2\,MB\,(L1 + L2) + MC\,L2 = -6\,A1\,\bar{x}1 / L1 - 6\,A2\,\bar{x}2 / L2$, where $A\,\bar{x}$ = moment of free BMD area about the outer supports.

Fixed end moments (FEM)

Load on span L	FEM at each end
Central point load W	$WL/8$
UDL w on full span	$wL^2/12$
Point load W at a from A, b from B	$MA = Wab^2/L^2$, $MB = Wa^2b/L^2$
Sinking of one support by Δ	$6EI\Delta/L^2$

Propped cantilever with UDL

- Prop reaction = $3wL/8$. Fixed end reaction = $5wL/8$. Fixed end moment = $wL^2/8$.
- Max positive moment = $9wL^2/128$ at $3L/8$ from the prop.

EXAMPLE Propped cantilever L = 4 m, w = 12 kN/m. Prop reaction R = 3 x 12 x 4/8 = 18 kN. Fixed end moment = 12 x 16/8 = 24 kN·m. Max positive BM = 9 x 12 x 16/128 = 13.5 kN·m at 1.5 m from the prop.

4. Displacement Methods

Slope deflection method

FORMULA $MAB = MFAB + (2EI/L)(2\theta A + \theta B - 3\Delta/L)$. Unknowns are joint rotations θ and sway Δ.

- Steps: write FEMs, write slope deflection equations, apply joint equilibrium ($\Sigma M = 0$ at each joint), solve for θ, back-substitute.

Moment distribution method (Hardy Cross)

- **Stiffness** of member (far end fixed) = $4EI/L$, carry-over factor = 1/2.
- Far end hinged: stiffness = $3EI/L$, carry-over factor = 0.
- **Distribution factor** DF = $k / \Sigma k$ at the joint. Sum of DFs at a joint = 1. DF for a fixed end = 0, for a free hinged end = 1.

EXAMPLE Beam ABC, A and C fixed, AB = BC = 4 m, same EI, UDL 12 kN/m on AB only. FEM AB = -12 x 16/12 = -16, FEM BA = +16 kN·m. DF at B = 0.5 each. Unbalanced 16, distribute -8 to BA and -8 to BC. Carry over -4 to A and -4 to C. Final: MAB = -20, MBA = +8, MBC = -8, MCB = -4 kN·m (clockwise positive).

- Sway frames: do one non-sway analysis and one sway analysis, then combine to satisfy the shear equation.

➤ Matrix methods

Flexibility (force) method	Stiffness (displacement) method
Unknowns: redundant forces	Unknowns: joint displacements
Size = Ds	Size = Dk
[δ] = [F][P]	[P] = [K][δ]
Hard to automate	Used in all software (STAAD, ETABS)

- Stiffness matrix is symmetric and its diagonal terms are positive. [K] = [F] inverse for the same coordinates.

FORMULA Beam element stiffness (EI/L^3) x [12, 6L, -12, 6L; 6L, $4L^2$, -6L, $2L^2$; -12, -6L, 12, -6L; 6L, $2L^2$, -6L, $4L^2$].

➤ Approximate methods for building frames

- **Vertical loads**: assume points of contraflexure at about 0.1 L from each end of a beam.
- **Portal method** (lateral loads): inflection points at mid-height of columns and mid-span of beams; interior columns take twice the shear of exterior columns.
- **Cantilever method** (tall frames): axial force in columns ∝ distance from the centroid of the column group.
- Used for preliminary design and checking computer output.

5. Arches and Cables

- **Three-hinged arch**: statically determinate (4 reactions, 3 equations + 1 at crown hinge).
- **Two-hinged arch**: indeterminate to 1st degree. Fixed arch: 3rd degree.
- For a three-hinged parabolic arch under full-span UDL, BM is zero everywhere (pure compression).

FORMULA Three-hinged parabolic arch, UDL w on full span L, rise h: $H = wL^2 / (8h)$, $V = wL/2$. Arch equation $y = 4h \times (L - x) / L^2$.

EXAMPLE Span 20 m, rise 4 m, UDL 10 kN/m. $H = 10 \times 20^2 / (8 \times 4) = 125$ kN. $V = 10 \times 20/2 = 100$ kN at each support. BM anywhere = 0.

- BM at any section of an arch = beam moment − H × y (H reduces the moment).
- **Cables**: carry only tension. Under UDL a cable takes a parabolic shape. Max tension at supports: $T = \sqrt{H^2 + V^2}$, $H = wL^2/(8d)$, d = sag.
- Temperature rise in a two-hinged arch reduces H.

6. Influence Lines and Rolling Loads

- **Influence line (IL)** = graph of a response (reaction, SF, BM) at a fixed point as a unit load moves across the structure.
- **Müller-Breslau principle**: the IL for a response is the deflected shape when that restraint is removed and a unit displacement is applied.
  - Gives exact shape for determinate structures (straight lines) and curved shapes for indeterminate ones.

➤ ILD for a simply supported beam (span L, section at a from A, b from B)

Response	Max ordinate	Location
Reaction RA	1	at A
SF at section	−a/L and +b/L	at the section (jump of 1)
BM at section	ab/L	at the section

- Response due to point loads = Σ (load × ordinate). Due to UDL = w × area of ILD under the load.
- **Max BM at a section** (train of loads): place loads so that average load on left = average load on right of the section.
- **Absolute max BM** (moving loads): occurs under a load when that load and the resultant are equidistant from mid-span.
- For a single moving point load, absolute max BM = WL/4 at mid-span.

REMEMBER UDL shorter than span: max BM at a section when the section divides the UDL in the same ratio as it divides the span.

7. Plastic Analysis

- **Plastic hinge**: section where full plastic moment Mp is reached; it rotates freely while holding Mp.
- **Shape factor** S = Mp / My = Zp / Ze.

Section	Shape factor
Rectangle	1.5
Solid circle	1.7 (16/(3π))
I-section	about 1.12 to 1.15
Diamond (square on diagonal)	2.0

- Collapse needs (Ds + 1) hinges for a complete mechanism.
- **Static theorem** (lower bound): safe load ≤ collapse load. **Kinematic theorem** (upper bound): load from any mechanism ≥ collapse load. **Uniqueness**: both equal at true collapse.

Beam	Collapse load
SSB, central point load	Wc = 4Mp/L
Fixed beam, central point load	Wc = 8Mp/L
Fixed beam, UDL	wc = 16Mp/L²
Propped cantilever, central point load	Wc = 6Mp/L
Propped cantilever, UDL	wc = 11.66Mp/L²

EXAMPLE Rectangle b = 100 mm, d = 200 mm, fy = 250 MPa. Zp = bd²/4 = 100 x 200²/4 = 10^6 mm³. Mp = 250 x 10^6 N·mm = 250 kN·m. Fixed beam span 5 m: Wc = 8 x 250/5 = 400 kN.

8. Most Asked Questions (practice)

1. Find Ds and Dk for given trusses and frames.
2. Find the deflection of a truss joint by the unit load method.
3. State and prove Maxwell's reciprocal theorem.
4. Analyse a continuous beam by the three moment theorem.
5. Analyse a portal frame (with and without sway) by slope deflection.
6. Analyse a continuous beam by moment distribution and draw the BMD.

7. Find H and BM in a three-hinged parabolic arch for a point load.
8. Draw ILD for reaction, SF and BM in a SSB; find max BM due to moving loads.
9. Use Müller-Breslau principle to sketch ILD of a propped cantilever.
10. Find the collapse load of a fixed beam and a portal frame by plastic analysis.

TIP In GATE, quick marks come from determinacy, FEMs, distribution factors, ILD ordinates, arch H, and shape factor. Do 5 moment distribution problems end to end before the exam.